Fourier-integral-operator approximation of solutions to first-order hyperbolic pseudodifferential equations I: convergence in Sobolev spaces

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Abstract

An approximation Ansatz for the operator solution, $U(z', z)$, of a hyperbolic first-order pseudodifferential equation, $\partial_z + a(z, x, D_x)$ with $\mathrm{Re}(a) \geq 0$, is constructed as the composition of global Fourier integral operators with complex phases. An estimate of the operator norm in $L(H^{(s)}, H^{(s)})$ of these operators is provided, which yields a convergence result for the Ansatz to $U(z', z)$ in some Sobolev space as the number of operators in the composition goes to ∞.

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0 Introduction

We consider the Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \leq Z \tag{0.1}$$

$$u\,|_{z=0} = u_0, \tag{0.2}$$

with $Z > 0$ and $a(z, x, \xi)$ continuous with respect to (w.r.t.) z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$ with the usual notation $D_x = \frac{1}{i}\partial_x$. Further assumptions will be made on the symbol $a(z, x, \xi)$. We denote by $U(z, 0)$ the solution operator of (0.1)–(0.2). When $a(z, x, \xi)$ is independent of x and z it is natural to treat such a problem by means of Fourier transformation:

$$u(z, x') = \iint \exp[i\langle x' - x|\xi\rangle - za(\xi)]\, u_0(x)\, đ\xi\, dx,$$

where $đ\xi := d\xi/(2\pi)^n$. For this to be well defined for all $u_0 \in \mathscr{S}(\mathbb{R}^n)$, we shall impose the real part of the principal symbol of a to be non-negative. When the symbol a depends on both x and z we can naively expect

$$u(z, x') \approx u_1(z, x') := \iint \exp[i\langle x' - x|\xi\rangle - za(0, x', \xi)]\, u_0(x)\, đ\xi\, dx$$

for z small and hence approximately solve the Cauchy problem (0.1)–(0.2) for $z \in [0, z^{(1)}]$ with $z^{(1)}$ small. If we want to progress in the z direction we have to solve the Cauchy problem

$$\begin{aligned} \partial_z u + a(z, x, D_x)u = 0, \quad z^{(1)} < z \le Z \\ u(z, .) \mid_{z=z^{(1)}} = u_1(z^{(1)}, .), \end{aligned}$$

which we again approximatively solve by

$$u(z, x') \approx u_2(z, x') := \iint \exp[i\langle x' - x|\xi\rangle - (z - z^{(1)})a(z^{(1)}, x', \xi)]\, u_1(z^{(1)}, x)\, đ\xi\, dx.$$

This procedure can be iterated until we reach $z = Z$.

If we denote by $\mathcal{G}_{(z',z)}$ the operator with kernel

$$G_{(z',z)}(x', x) = \int \exp[i\langle x' - x|\xi\rangle] \exp[-(z' - z)a(z, x', \xi)]\, đ\xi,$$

then combining all iteration steps above involves composition of such operators: let $0 \le z^{(1)} \le \cdots \le z^{(k)} \le Z$, we then have

$$u_{k+1}(z, x) = \mathcal{G}_{(z,z^{(k)})} \circ \mathcal{G}_{(z^{(k)},z^{(k-1)})} \circ \cdots \circ \mathcal{G}_{(z^{(1)},0)}(u_0)(x),$$

when $z \ge z^{(k)}$. We then define the operator $\mathcal{W}_{\mathfrak{P},z}$ for a subdivision $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$, of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$,

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \le z \le z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})} \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \le z \le z^{(k+1)}. \end{cases}$$

According to the procedure described above, $\mathcal{W}_{\mathfrak{P},z}(u_0)$ yields an approximation Ansatz for the solution to the Cauchy problem (0.1)–(0.2) with step size $\Delta_{\mathfrak{P}} = \sup_{i=1,\ldots,N}(z_i - z_{i-1})$. The operator $\mathcal{G}_{(z',z)}$ is often referred to as the *thin-slab propagator* (see e.g. [3, 2]).

Note that a similar procedure can be used to show the existence of an evolution system by approximating it by composition of semigroup solutions of the Cauchy problem with z 'frozen' in $a(z, x, D_x)$ [11, 19]. Note that the thin-slab propagator $\mathcal{G}_{(z',z)}$ is however not a semigroup nor an evolution family here (see Section 3 for simple arguments).

The approximation Ansatz proposed here is a tool to compute approximations of the exact solution to the Cauchy problem (0.1)–(0.2). Such computations in applications to geophysical problems have been used in [3]. In exploration seismology one is confronted with solving equations of the type

$$(\partial_z - ib(z, x, D_t, D_x) + c(z, x, D_t, D_x))v = 0, \tag{0.3}$$

$$v(0, .) = v_0(.), \tag{0.4}$$

where t is time, z is the vertical coordinate and x is the lateral or transverse coordinate. The operators b and c are of first order, with real principal parts, b_1 and c_1, where $c_1(z, x, \tau, \xi)$ is non-negative. Note that the Cauchy problem (0.1)–(0.2) studied here is more general. The Cauchy problem (0.3)–(0.4) is obtained by a (microlocal) decoupling of the up-going and down-going wavefields in the acoustic wave equation (see Appendix A and [21] for details). In practice, the proposed Ansatz can then be a tool to approximate the exact solution for the purpose of imaging the Earth's interior [3, 2]. As explained in Appendix A the operator c acts as a damping term that suppresses singularities in the microlocal region where its symbol does not vanishes. This effect is recovered in the proposed Ansatz. Seismic imaging aims at recovering the singularities in the subsurface (see for instance [23, 1]). Thus, geophysists are not only interested in the convergence of this Ansatz to the exact solution of the Cauchy problem (0.3)–(0.4) but they expect the wavefront set of the approximate solution to be close, in some sense, to that of the exact solution. We shall investigate the microlocal properties of the proposed Ansatz in Part II, written in collaboration with Günther Hörmann.

In the present article, we are interested in the analysis of the convergence of the approximation scheme $\mathcal{W}_{\mathfrak{P}}$ in Sobolev spaces. Section 1 introduces the Cauchy problem we study and the precise assumptions made on the symbol $a(z, x, \xi)$, especially on the real part, c_1, and imaginary part, $-b_1$, of its principal symbol. In Section 2, we shall at first concentrate our study on the operator $\mathcal{G}_{(z',z)}$, yet to be properly defined. Under some assumptions on $a(z, x, \xi)$, we shall prove that $\mathcal{G}_{(z',z)}$ is a global Fourier integral operator (FIO) with complex phase and that it maps $\mathscr{S}$ into $\mathscr{S}$, $\mathscr{S}'$ into $\mathscr{S}'$ and $H^{(s)}$ into $H^{(s)}$ for any s. An estimation of $\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})}$ will be the first step towards the analysis in Section 3 of the convergence of $\mathcal{W}_{\mathfrak{P},z}$. In fact we prove that for $z' - z$ sufficiently small then (Theorem 2.26)

$$\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})} \leq 1 + M|z' - z|,$$

for some constant M. Such an estimate is achieved by the analysis of the behavior of the symbol $\exp[-\Delta c_1]$ as an element of $S^{0}_{\frac{1}{2}}$, in particular as $\Delta = z' - z$ goes to zero.

In Section 3 we study the convergence of the Ansatz $\mathcal{W}_{\mathfrak{P},z}(u_0)$ to the solution of the Cauchy problem (0.1)–(0.2) in Sobolev spaces as $\Delta_{\mathfrak{P}}$ goes to 0. A convergence in norm of $\mathcal{W}_{\mathfrak{P},z}$ to the solution operator of the Cauchy problem (0.1)–(0.2) is actually obtained (Theorem 3.11):

$$\lim_{\Delta_{\mathfrak{P}} \to 0} \|\mathcal{W}_{\mathfrak{P},z} - U(z, 0)\|_{(H^{(s+1)},H^{(s)})} = 0,$$

with a convergence rate of order $\frac{1}{2}$ when $a(z, .)$ is in $\mathscr{C}^{0,\alpha}$ w.r.t. z, $\alpha \geq \frac{1}{2}$. We furthermore obtain (Theorem 3.18)

$$\lim_{\Delta_{\mathfrak{P}} \to 0} \|\mathcal{W}_{\mathfrak{P},z} - U(z, 0)\|_{(H^{(s+1)},H^{(s+r)})} = 0, \quad 0 \leq r < 1,$$

with a convergence rate of order $(1 - r)/2$ while the operator $\mathcal{W}_{\mathfrak{P},z}$ strongly converges to $U(z, 0)$ in $H^{(s+1)}$.

At the end Section 3 we relax some regularity property of the symbol $a(z, .)$ w.r.t. z by the introduction of another, yet natural, Ansatz: following [17], the thin-slab propagator, $\mathcal{G}_{(z',z)}$, is replaced by the operator $\widehat{\mathcal{G}}_{(z',z)}$ with kernel

$$\widehat{G}_{(z',z)}(x', x) = \int \exp[i\langle x' - x|\xi\rangle] \exp[-\textstyle\int_z^{z'} a(s, x', \xi) ds]\, đ\xi.$$

In Part II, we shall focus on the microlocal aspects of the operator $\mathcal{W}_{\mathfrak{P},z}$ and how it propagates the singularities of the initial condition u_0. We shall show that the wavefront set of $\mathcal{W}_{\mathfrak{P},z}(u_0)(z, .)$ converges in some sense to that of the solution $u(z, .)$ of the Cauchy problem (0.1)–(0.2) as $\Delta_{\mathfrak{P}}$ goes to 0.

Multi-composition of FIOs to approximate solutions of Cauchy problems where first proposed in [16] and [15]. In these articles the exact solution operator of a first order hyperbolic system is approximated with a different Ansatz. The approximation is made up to a regularizing operator. The technique is based on the computation and the estimation of the phase functions and amplitudes of the FIO resulting from these multi-products, a result know as the Kumano-go-Taniguchi Theorem. The technique was then further applied to Schrödinger equations with specific symbols [12, 17]. In these latter works, the multi-product in also interpreted as an iterated integral of Feynman's type and convergence is studied in a weak sense. In [12] a convergence result in L^2 is proven. This is the type of results sought here for first order hyperbolic equations. We however do not use the apparatus of multi-phases and rather focus on estimating the Sobolev regularity of each term in the multi-product of FIOs in the proposed Ansatz. While the resulting product is an FIO, we do not compute its phase and amplitude. The Sobolev regularity allows us to use a priori energy estimates for the Cauchy problem (0.1)–(0.2) to prove convergence of the approximating Ansatz to the solution operator.

In this article, when the constant C is used, its value may change from one line to the other. If we want to keep track of the value of a constant we shall use another letter. When we write that a function is bounded w.r.t. z and/or Δ we shall actually mean that z is to be taken in the interval $[0, Z]$ and Δ in some interval $[0, \Delta_{max}]$ unless otherwise stipulated. We shall generally write X, X', X'', $X^{(1)}$, $\dots$, $X^{(N)}$ for $\mathbb{R}^n$, according to variables, e.g., x, x', $\dots$, $x^{(N)}$.

In a standard way, we set $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^p$. Throughout the article, we use spaces of global symbols; a function $a \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ is in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$, $0 < \rho \leq 1$, $0 \leq \delta < 1$, if for all multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\,(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^p.$$

The best possible constants $C_{\alpha\beta}$, i.e.,

$$p_{\alpha\beta}(a) := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^p} (1+|\xi|)^{-m+\rho|\beta|-\delta|\alpha|}|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|,$$

define seminorms for a Fréchet space structure on $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^p)$. As usual we write $S^m_\rho(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1-\delta$, $\frac{1}{2} \leq \rho < 1$, and $S^m(\mathbb{R}^n \times \mathbb{R}^p)$ in the case $\rho = 1$, $\delta = 0$.

We shall use, in a standard way, the notation # for the composition of symbols of pseudodifferential operators (ψDO). When given an amplitude $p(x,y,\xi) \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$, $\rho \geq \delta$, we shall also use the notation $\sigma\{p\}(x,\xi)$ for the symbol of the pseudodifferential operator with amplitude p. For $p \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ we shall write p^* for the symbol of the adjoint operator. When composing ψDOs or computing adjoints of ψDOs we shall make use of the oscillatory integral representation of the resulting symbol instead of asymptotic series for two reasons. First, we aim at estimating operator norm in $L(H^s, H^s)$ while using asymptotic series representations for symbols yields results up to regularizing operators which norm cannot be estimated. Second, we shall consider symbols in S^m_ρ, for some m, including the case $\rho = \frac{1}{2}$ for which the asymptotic formulae of the calculus of ψDOs cease to hold.

For $r \in \mathbb{R}$ we let $E^{(r)}$ be the ψDO with symbol $\langle\xi\rangle^r$. The operator $E^{(r)}$ maps $H^{(s)}(X)$ onto $H^{(s-r)}(X)$ unitarily for all $s \in \mathbb{R}$ with $E^{(-r)}$ being the inverse map.

1 The homogeneous first-order hyperbolic equation

Let $s \in \mathbb{R}$ and $Z > 0$. We consider the Cauchy problem

$$\partial_z u + a(z, x, D_x)u = 0, \quad 0 < z \leq Z, \tag{1.5}$$

$$u \,|_{z=0} = u_0 \in H^{(s+1)}(\mathbb{R}^n), \tag{1.6}$$

where the symbol $a(z, x, \xi)$ satisfies the following assumption.

Assumption 1.1.

$$a_z(x, \xi) = a(z, x, \xi) = -i\, b(z, x, \xi) + c(z, x, \xi),$$

where $b \in \mathscr{C}^0([0, Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, with real principal symbol b_1 homogeneous of degree 1 for $|\xi|$ large enough and $c \in \mathscr{C}^0([0, Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$ with non-negative principal symbol c_1 homogeneous of degree 1 for $|\xi|$ large enough. Without loss of generality we can assume that b_1 and c_1 are homogeneous of degree 1 for $|\xi| \geq 1$.

In Section 3 we shall further make the following assumption.

Assumption 1.2. *The symbol $a(z, .)$ is assumed to be in $\mathscr{L}([0, Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, i.e. Lipschitz continuous w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$, in the sense that,*

$$a(z', x, \xi) - a(z, x, \xi) = (z' - z)\tilde{a}(z', z, x, \xi), \quad 0 \leq z \leq z' \leq Z$$

with $\tilde{a}(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$.

The case of Hölder continuity will also be addressed. Weaker assumptions will also be formulated in Section 3, for instance by the introduction of another approximating Ansatz.

We denote by $a_1 = -ib_1 + c_1$ the principal symbol of a and write $b = b_1 + b_0$ with $b_0 \in \mathscr{C}^0([0, Z], S^0(\mathbb{R}^n \times \mathbb{R}^n))$ and $c = c_1 + c_0$ with $c_0 \in \mathscr{C}^0([0, Z], S^0(\mathbb{R}^n \times \mathbb{R}^n))$. Assumption 1.1 ensures that the hypotheses (i)–(iii) of Theorem 23.1.2 in [8] are satisfied. Then there exists a unique solution in $\mathscr{C}^0([0, Z], H^{(s+1)}(\mathbb{R}^n)) \cap \mathscr{C}^1([0, Z], H^{(s)}(\mathbb{R}^n))$ to the Cauchy problem (1.5)–(1.6).

Furthermore, we have the following energy estimate [8, Lemma 23.1.1] for any function in $\mathscr{C}^1([0, Z], H^{(s)}(\mathbb{R}^n)) \cap \mathscr{C}^0([0, Z], H^{(s+1)}(\mathbb{R}^n))$

$$\sup_{z \in [0,Z]} \exp[-\lambda z]\, \|u(z, .)\|_{H^{(s)}} \leq \|u(0, .)\|_{H^{(s)}} + 2 \int_0^Z \exp[-\lambda z]\, \|\partial_z u + a_z(x, D_x)u\|_{H^{(s)}} dz, \tag{1.7}$$

with λ large enough (λ solely depending on s).

By Proposition 9.3 in [5, Chapter VI] the family of operators $(a_z)_{z \in [0,Z]}$ generates a strongly continuous evolution system. Let $U(z', z)$ denote the corresponding evolution system:

$$U(z'', z') \circ U(z', z) = U(z'', z), \quad Z \geq z'' \geq z' \geq z \geq 0,$$

with

$$\partial_z U(z,z_0)u_0 + a(z,x,D_x)U(z,z_0)u_0 = 0, \quad 0 \leq z_0 < z \leq Z,$$
$$U(z_0,z_0)u_0 = u_0 \in H^{(s+1)}(\mathbb{R}^n),$$

while $U(z,z_0)u_0 \in H^{(s+1)}(\mathbb{R}^n)$ for all $z \in [z_0, Z]$. For the Cauchy problem (1.5)–(1.6) we take $z_0 = 0$.

2 The thin-slab propagator. Regularity properties.

We follow the terminology introduced in [9, Sections 25.4-5] for FIOs with complex phase. Let $z', z \in [0, Z]$ with $z' \geq z$ and let $\Delta := z' - z$. Define $\phi_{(z',z)} \in \mathscr{C}^\infty(X' \times X \times \mathbb{R}^n)$ as

$$\begin{aligned}\text{(2.8)} \quad \phi_{(z',z)}(x', x, \xi) &:= \langle x' - x|\xi\rangle + i\Delta a_1(z, x', \xi) \\ &= \langle x' - x|\xi\rangle + \Delta b_1(z, x', \xi) + i\Delta c_1(z, x', \xi).\end{aligned}$$

Remark 2.1. The function $\phi_{(z',z)}$ is assumed to be homogeneous of degree one only when $|\xi| \geq 1$. This however is not an obstacle to the subsequent analysis, e.g., FIO properties, since to define such operators the phase function need not be homogeneous of degree one for small $|\xi|$. In the subsequent results concerning the phase function and FIOs one will then assume that $|\xi|$ is large enough, i.e., $|\xi| \geq 1$.

Lemma 2.2. *$\phi_{(z',z)}$ is a nondegenerate complex phase function of positive type (at any point (x'_0, x_0, ξ_0) where $\partial_\xi \phi_{(z',z)} = 0$).*

Proof. Note that, by Assumption 1.1, $\mathrm{Im}(\phi_{(z',z)}) \geq 0$ and $\phi_{(z',z)}$ is homogeneous of degree one; $\partial_x \phi_{(z',z)} = 0$ implies $\xi = 0$. Thus, $\phi_{(z',z)}$ is a phase function of positive type. Inspecting the partial derivatives of $\partial_\xi \phi_{(z',z)}$ w.r.t. x we conclude that the differentials $d(\partial_{\xi_1}\phi_{(z',z)}), \dots, d(\partial_{\xi_n}\phi_{(z',z)})$ are linearly independent. ■

With $a_0(z, .) \in S^0(X \times \mathbb{R}^n)$ we have $\exp[-\Delta a_0(z, .)] \in S^0(X \times \mathbb{R}^n)$ by Lemma 18.1.10 in [8]. We define

$$\text{(2.9)} \qquad g_{(z',z)}(x, \xi) := \exp[-\Delta a_0(z, x, \xi)].$$

We shall keep this notation (for this symbol and others in the sequel) but it will be useful however to consider this symbol to depend on the parameters z and Δ instead of z and z' in the following analysis. Note that $g_{(z',z)}$ is bounded w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^0(X \times \mathbb{R}^n)$. Hence, we may define a distribution kernel $G_{(z',z)}(x', x) \in \mathscr{D}'(X' \times X)$

$$\begin{aligned}G_{(z',z)}(x', x) &= \int \exp[i\langle x' - x|\xi\rangle] \exp[-\Delta a(z, x', \xi)]\, đ\xi \\ &= \int \exp[i\phi_{(z',z)}(x', x, \xi)]\, g_{(z',z)}(x', \xi)\, đ\xi\end{aligned}$$

as an oscillatory integral. We denote the associated operator by $\mathcal{G}_{(z',z)}$. This operator is often referred to as the *thin-slab propagator* (see e.g. [3, 2]). We show that $\mathcal{G}_{(z',z)}$ is a global FIO in $\mathbb{R}^n$.

Define $\alpha := (x', x, \xi', \xi)$ and

$$\begin{aligned}
u_{\theta_j}(\alpha,\theta) &= \partial_{x_j}\phi_{(z',z)}(x', x, \theta) + \xi_j = -\theta_j + \xi_j,\\
u_{\xi_j}(\alpha,\theta) &= \partial_{x'_j}\phi_{(z',z)}(x', x, \theta) - \xi'_j = \theta_j - \xi'_j + i\Delta\partial_{x_j}a_1(z, x', \theta),\\
u_{x_j}(\alpha,\theta) &= \partial_{\theta_j}\phi_{(z',z)}(x', x, \theta) = x'_j - x_j + i\Delta\partial_{\xi_j}a_1(z, x', \theta),
\end{aligned}$$

where $j = 1, \ldots, n$. We denote by $\hat{J}_{(z',z)}$ the ideal in $\mathscr{C}^\infty(\mathbb{R}^{5n})$ generated by the functions u_{θ_j}, u_{ξ_j}, and u_{x_j}, and we let $J_{(z',z)}$ be the subset of the functions in $\hat{J}_{(z',z)}$ that are independent of θ.

Lemma 2.3. *There exists $\Delta_1 > 0$, such that, for all $z', z \in [0, Z]$, with $z' > z$ and $\Delta = z' - z \le \Delta_1$, the ideal $J_{(z',z)}$ is generated by the functions*

$$\begin{aligned}
v_{\xi_j}(\alpha) &= \partial_{x'_j}\phi_{(z',z)}(x', x, \xi) - \xi'_j\\
&= \xi_j - \xi'_j + i\Delta\partial_{x_j}a_1(z, x', \xi) = u_{\xi_j}|_{\theta=\xi},\\
v_{x_j}(\alpha) &= \partial_{\xi_j}\phi_{(z',z)}(x', x, \xi) = x'_j - x_j + i\Delta\partial_{\xi_j}a_1(z, x', \xi) = u_{x_j}|_{\theta=\xi}
\end{aligned} \tag{2.10}$$

$j = 1, \ldots, n$.

Some of the key arguments of the proof are close to that in the proof of Theorem 25.4.4 in [9].

Proof. The ideal $\hat{J}_{(z',z)}$ is also generated by the functions

$$u_{\theta_j},\quad \tilde{u}_{\xi_j} := u_{\theta_j} + u_{\xi_j} = \xi_j - \xi'_j + i\Delta\partial_{x_j}a_1(z, x', \theta),\quad u_{x_j},$$

$j = 1, \ldots, n$. We define $\nu := (x', \xi', \theta)$, $\mu := (x, \xi)$. We set a point $(\nu_0, \mu_0) = (x'_0, \xi'_0, \theta_0, x_0, \xi_0)$ where these generators vanish and we work in a neighborhood of this point. (Note that $\theta_0 = \xi_0$.) Since $z \mapsto a_1(z, .) \in S^1(X \times \mathbb{R}^n)$ is bounded we have that $\exists\Delta_1 > 0$ such that for $0 \le \Delta \le \Delta_1$, and all $z \in [0, Z]$,

$$\det \partial\left(u_{\theta_1}, \ldots, u_{\theta_n}, \tilde{u}_{\xi_1}, \ldots, \tilde{u}_{\xi_n}, u_{x_1}, \ldots, u_{x_n}\right)/\partial\nu \neq 0$$

and

$$\det \partial\left(v_{\xi_1}, \ldots, v_{\xi_n}, v_{x_1}, \ldots, v_{x_n}\right)/\partial(x', \xi') \neq 0.$$

By Theorem 7.5.7 in [10] we have

$$\begin{pmatrix} x' - x\\ \xi' - \xi\\ \theta \end{pmatrix} = \begin{pmatrix} Q(\nu,\mu) & P(\nu,\mu)\\ 0 & I_n \end{pmatrix} \begin{pmatrix} u_x\\ \tilde{u}_\xi\\ -u_\theta \end{pmatrix} + \begin{pmatrix} \tilde{x}(\mu)\\ \tilde{\xi}(\mu)\\ \xi \end{pmatrix},$$

where P is a $\mathscr{C}^\infty$ $2n \times n$ matrix and Q is a $\mathscr{C}^\infty$ $2n \times 2n$ matrix and the functions $\tilde{x}$, $\tilde{\xi}$ are also $\mathscr{C}^\infty$ in a neighborhood of (ν_0, μ_0). As the functions $w_x(\nu,\mu) := x' - x - \tilde{x}(\mu)$, $w_\xi(\nu,\mu) := \xi' - \xi - \tilde{\xi}(\mu)$, $w_\theta(\nu,\mu) := \theta - \xi$ have linearly independent differentials, Lemma 7.5.8 in [10] proves that they generate $\hat{J}_{(z',z)}$ and the proof of that lemma shows that Q is invertible in a neighborhood of (ν_0, μ_0). Letting $\theta = \xi$ we have

$$Q(x', \xi', \theta = \xi, x, \xi)^{-1}\begin{pmatrix} w_x(\nu,\mu)\\ w_\xi(\nu,\mu) \end{pmatrix} = \begin{pmatrix} u_x(x', x, \xi)\\ \tilde{u}_\xi(x', x, \xi) \end{pmatrix} = \begin{pmatrix} v_x(\alpha)\\ v_\xi(\alpha) \end{pmatrix}.$$

We thus obtained that $\hat{J}_{(z',z)}$ is generated by the functions u_{θ_j}, v_{x_j}, v_{ξ_j}, $j = 1, \dots, n$. We then see that $J_{(z',z)}$ is generated by v_{x_j}, v_{ξ_j}, $j = 1, \dots, n$. In fact, using Theorem 7.5.7 in [10] again, any $\mathscr{C}^\infty$ function $h(\alpha)$ can be locally written in the form

$$h(\alpha) = \sum_{1 \le i \le n} (h_{x_j}(\alpha', \mu) v_{x_j}(\alpha', \mu) + h_{\xi_j}(\alpha', \mu) v_{\xi_j}(\alpha', \mu)) + r(\mu),$$

with $\alpha' = (x', \xi')$ provided that $0 \le \Delta \le \Delta_1$. If $h \in J_{(z',z)}$ then $r \in J_{(z',z)}$ and Lemma 7.5.10 in [10] implies that $\forall N \in \mathbb{N}$, $\exists C_N > 0$:

$$r(\mu) \le C_N \max(|\mathrm{Im}\ \tilde{x}(\mu)|, |\mathrm{Im}\ \tilde{\xi}(\mu)|)^N,$$

locally. Therefore, Theorem 7.5.12 in [10] yields $r \in I(w_x, w_\xi) = I(v_x, v_\xi)$; which in turn implies $h \in I(v_x, v_\xi)$ and thereby completes the proof. ■

As the Poisson brackets (for the symplectic 2-form $\sigma' - \sigma$ on $T^*(X' \times X)$, where σ' and σ are the symplectic 2-forms on $T^*(X')$ and $T^*(X)$ respectively) of any two of the functions in (2.10) vanish identically we obtain that the ideal generated by these functions is globally a conic canonical ideal in the sense of [9, Definition 25.4.1. and Section 25.5]. The phase function $\phi_{(z',z)}$ thus defines $J_{(z',z)}$ in the neighborhood of any point of $J_{(z',z)\mathbb{R}}$: it thus *globally* defines $J_{(z',z)}$, which is then of positive type. Therefore the operator $\mathcal{G}_{(z',z)}$ is a global FIO with complex phase (see Definitions 25.4.9. and 25.5.1. in [9]).

Proposition 2.4. *There exists $\Delta_1 > 0$ such that if $0 \le \Delta = z' - z \le \Delta_1$ then the operator $\mathcal{G}_{(z',z)}$ is a global Fourier integral operator with complex phase and $G_{(z',z)} \in I^0(X' \times X, (J_{(z',z)})', \Omega^{1/2}_{X' \times X})$.*

We denote the half density bundle on $X' \times X$ by $\Omega^{1/2}_{X' \times X}$. Note that $(J_{(z',z)})'$ stands for the twisted canonical ideal, i.e. a Lagrangian ideal (see Section 25.5 in [9]).

Note that, with the following analysis, we could also consider $\mathcal{G}_{(z',z)}$ as a global FIO with real phase with amplitude in $S^0_{\frac{1}{2}}(X' \times X \times \mathbb{R}^n)$ (see e.g. [20]). However such consideration would be rather technical as one usually restricts oneself to the type S^m_ρ with $\rho > \frac{1}{2}$ for FIOs (see the remark at the end of Section 25.1 in [9]; see also [18, pages 391-392]). Viewing the thin-slab propagator $\mathcal{G}_{(z',z)}$ as a FIO with complex phase is also a good framework to understand the propagation of singularities in Part II. We shall however make this interpretation for $\mathcal{G}_{(z',z)}$ in Proposition 2.29, below, to apply a result of Kumano-go [13, Theorem 2.5].

We now establish some global continuity properties of the operator $\mathcal{G}_{(z',z)}$ stated in a slightly more general form (for similar results with global symbols see for instance [13], where phase functions are real and other conditions are imposed on the phase function).

Lemma 2.5. *Let A be an FIO with a kernel of the form*

$$K_A(x, y) = \int \exp[i\varphi(x, \xi) - i\langle y|\xi\rangle]\, \sigma_A(x, \xi) đ\xi \in \mathscr{D}'(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\sigma_A \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $Im(\varphi(x, \xi)) \ge 0$ and φ is homogeneous of degree one in ξ, for $|\xi|$ large enough, and $\partial_{x_i}\varphi \in S^1(\mathbb{R}^n \times \mathbb{R}^n)$. Furthermore, for all $i = 1, \dots, n$ we assume $\partial_{\xi_i}\varphi(x, \xi) = x_i + f_i(x, \xi)$ where $f_i \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$. Then A maps $\mathscr{S}$ into $\mathscr{S}$ continuously.

Proof. Let $u \in \mathscr{S}$. We then have

$$
\begin{aligned}
|Au(x)| &\le \int |\sigma_A(x,\xi)(1+|\xi|)^{-m}||(1+|\xi|)^m \hat{u}(\xi)|\, đ\xi \\
&\le C \sup_{\xi\in\mathbb{R}^n} |\sigma_A(x,\xi)(1+|\xi|)^{-m}| \sup_{\xi\in\mathbb{R}^n} |(1+|\xi|)^{m+n+1}\hat{u}(\xi)|,
\end{aligned}
$$

where $C = \int (1+|\xi|)^{-n-1} đ\xi$. The operator A is hence well defined from $\mathscr{S}$ into $\mathscr{C}(\mathbb{R}^n)$. If we differentiate we obtain

$$
D_{x_i} Au(x) = \int \exp[i\varphi(x,\xi)]\, (\partial_{x_i}\varphi(x,\xi)\sigma_A(x,\xi) - i\partial_{x_i}\sigma_A(x,\xi))\, \hat{u}(\xi)\, đ\xi\,.
$$

Noting that $\partial_{x_i}\varphi(x,\xi)\sigma_A(x,\xi) - i\partial_{x_i}\sigma_A(x,\xi) \in S^{m+1}(\mathbb{R}^n \times \mathbb{R}^n)$ we similarly have

$$
\begin{aligned}
|D_{x_i} Au(x)| &\le C \sup_{\xi\in\mathbb{R}^n} |(1+|\xi|)^{m+n+2}\hat{u}(\xi)| \\
&\le C' \sup_{x\in\mathbb{R}^n} |x^\alpha D_x^\beta u(x)| \ \text{ for some } \alpha,\beta \ge 0.
\end{aligned}
$$

Iterating we find that $Au \in \mathscr{C}^\infty(\mathbb{R}^n)$. Integrating by parts we also have

$$
\begin{aligned}
A(x_j u)(x) &= \int \exp[i\varphi(x,\xi)] \left(\partial_{\xi_i}\varphi(x,\xi)\sigma_A(x,\xi) - i\partial_{\xi_i}\sigma_A(x,\xi)\right) \hat{u}(\xi)\, đ\xi \\
&= x_j Au(x) + \int \exp[i\varphi(x,\xi)] \left(f_i(x,\xi)\sigma_A(x,\xi) - i\partial_{\xi_i}\sigma_A(x,\xi)\right) \hat{u}(\xi)\, đ\xi.
\end{aligned}
$$

Since $f_i(x,\xi)\sigma_A(x,\xi) - i\partial_{\xi_i}\sigma_A(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ we obtain

$$
|x_j Au(x)| \le C \sup_{x\in\mathbb{R}^n} |x^\alpha D_x^\beta u(x)| + C \sup_{x\in\mathbb{R}^n} |x^{\alpha'} D_x^{\beta'} u(x)|,
$$

for some $\alpha, \alpha', \beta, \beta' \ge 0$. Similar estimates hold for $|x^\alpha D_x^\beta Au(x)|$ because of the hypothesis made on f_i, $i = 1, \ldots, n$. The operator A thus maps $\mathscr{S}$ into $\mathscr{S}$ continuously. ∎

To show continuity from $\mathscr{S}'$ into $\mathscr{S}'$ we shall need the following lemma.

Lemma 2.6. *Let j,k non-negative integers, $u \in \mathscr{S}(\mathbb{R}^n)$, $f \in \mathscr{C}^{k+1}(\mathbb{R}^n)$ such that*

$$
0 \le \mathit{Im}\ f(x) \le C_0,\ x \in \mathbb{R}^n, \quad |f^{(r)}(x)| \le C_r,\ x \in \mathbb{R}^n,\ 1 \le r \le k+1.
$$

Then we have

$$
\begin{aligned}
\omega^{j+k} \left| \int u(x)(\mathit{Im}\ f(x))^j \exp[i\omega f(x)]\, dx \right| \\
\le C \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha u(x)| (|f'(x)|^2 + \mathit{Im}\ f(x))^{|\alpha|/2-k}, \ \ \omega > 0,
\end{aligned}
\tag{2.11}
$$

where the constant C is bounded when the function f stays in a domain of $\mathscr{C}^{k+1}(\mathbb{R}^n)$ where $C_0, C_1, \ldots, C_{k+1}$ can be chosen bounded.

Proof. The proof is the same as that of Theorem 7.7.1 in [10] where $u \in \mathscr{C}_0^k(\mathbb{R}^n)$. In fact the further assumptions on f made here allow to give global bounds that are needed since $u \in \mathscr{S}$ in the present case. ∎

Lemma 2.7. *Let A be an FIO with a kernel of the form*

$$K_A(x,y) = \int \exp[i\langle x-y|\xi\rangle + i\gamma(x,\xi)]\,\sigma_A(x,\xi)đ\xi \in \mathscr{D}'(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\sigma_A \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\gamma \in S^1(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $Im(\gamma(x,\xi)) \geq 0$, and γ is homogeneous of degree one in ξ, for $|\xi|$ large enough. Furthermore, we assume that there exists $d \geq 0$ such that

$$|Re\,(\partial_x\gamma(x,\xi))\,| \leq d < 1,\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ |\xi| = 1. \tag{2.12}$$

Then A maps $\mathscr{S}'$ into $\mathscr{S}'$ continuously.

Observe that the differential of $\phi(x,\xi) := \langle x-y|\xi\rangle + \gamma(x,\xi)$ does not vanish in $\mathbb{R}^{2n}\times\mathbb{R}^n\backslash 0$. The function ϕ is thus a complex phase function. The differentials $d(\partial_{\xi_1}\phi),\ldots,d(\partial_{\xi_n}\phi)$ are linearly independent. Hence ϕ is a nondegenerate complex phase function of positive type. Note that by (2.12), the function $\langle x-y|\xi\rangle + \gamma(x,\xi)$ is an operator phase function in the sense of [6, Definition 1.4.4.].

Proof. Without loss of generality we may assume that γ is homogeneous of degree one for $|\xi| \geq 1$. Let A^t be the transpose of A and let $u \in \mathscr{S}$, then.

$$A^t u(x) = \int \exp[-i\langle x|\xi\rangle] \int \exp[i\langle y|\xi\rangle + i\gamma(y,\xi)]\,\sigma_A(y,\xi)\,u(y)\,dyđ\xi$$

Define

$$v(\xi,\eta) = \int \exp[i\langle y|\xi\rangle + i\gamma(y,\xi)]\,\sigma_A(y,\eta)\,u(y)\,dy\,,$$

and put $w(\xi) = v(\xi,\xi)$. As $u \in \mathscr{S}$ then v and w are both $\mathscr{C}^\infty$. Then $A^t u$ is the Fourier transform of w. The lemma is proven if we show that $u \mapsto w(\xi)$ is continuous from $\mathscr{S}$ to $\mathscr{S}$.

Let $\omega = |\xi| \geq 1$ and $\xi_0 = \xi/|\xi| \in \mathcal{S}^{n-1}$. We then have $\langle y|\xi\rangle + \gamma(y,\xi) = \omega f(y,\xi)$ with f homogeneous of degree zero in ξ, for $|\xi| \geq 1$. Note that $\partial_y f(y,\xi) = \xi_0 + \partial_y\gamma(y,\xi_0)$. With the assumption made on $\partial_y\gamma$ we have $|\partial_y f(y,\xi)| \geq c > 0$. Applying Lemma 2.6 and estimate (2.11) we obtain

$$\begin{aligned}\omega^k|v(\xi,\eta)| &\leq K_k \sum_{|\alpha|\leq k} \sup |D_y^\alpha(\sigma_A(y,\eta)u(y))| \\ &\leq K_k'(1+|\eta|)^m \sup_{\substack{|\alpha|\leq k\\ y\in\mathbb{R}^n}} |D^\alpha u(y)|,\ \omega \geq 1\end{aligned}$$

where the constants K_k, K_k' can be chosen uniformly w.r.t. ξ, $|\xi| \geq 1$ since the constants $C_0, C_1, \ldots, C_{k+1}$ of Lemma 2.6 can be chosen bounded (as $\xi_0 \in \mathcal{S}^{n-1}$). Now setting $\eta = \xi$ we obtain that for all $k \in \mathbb{N}$, $\exists K_k'' > 0$

$$(1+|\xi|)^{k-m}|w(\xi)| \leq K_k'' \sup_{\substack{|\alpha|\leq k\\ y\in\mathbb{R}^n}} |D^\alpha u(y)|\,,\ \xi \in \mathbb{R}^n,\ |\xi| \geq 1. \tag{2.13}$$

We now consider

$$D_{\xi_i} w(\xi) = \int \exp[i\langle y|\xi\rangle + i\gamma(y,\xi)]((y_i + \partial_{\xi_i}\gamma(y,\xi))\sigma_A(y,\xi) - i\partial_{\xi_i}\sigma_A(y,\xi))u(y)dy\,.$$

As $y_i u(y) \in \mathscr{S}$ and $\partial_{\xi_i}\gamma(y,\xi)$ is homogeneous of degree zero for $|\xi| \geq 1$ estimates similar to those in (2.13) are valid. ■

It is immediate from the structure of $\phi_{(z',z)}$ in (2.8) that Lemma 2.5 applies to $\mathcal{G}_{(z',z)}$. If $\Delta = z' - z$ is small enough we have $|\Delta \partial_{x_i} b_1(z, x', \xi)| \leq d < 1$, due to Assumption 1.1, and then Lemma 2.7 applies. We thus have the following proposition.

Proposition 2.8. *There exists $\Delta_2 > 0$ such that if $z', z \in [0, Z]$ with $0 \leq \Delta := z' - z \leq \Delta_2$ then $\mathcal{G}_{(z',z)}$ maps $\mathscr{S}$ into $\mathscr{S}$ and $\mathscr{S}'$ into $\mathscr{S}'$ continuously.*

Remark 2.9. By the above result, composition of the two FIOs $\mathcal{G}_{(z'',z')}$ and $\mathcal{G}_{(z',z)}$ is thus natural without further requirement such as having the operators properly supported.

We now turn to *global* L^2 and Sobolev space continuity for the operator $\mathcal{G}_{(z',z)}$. We shall use the following lemma.

Lemma 2.10. *Let $p_s(y, \eta)$ be bounded w.r.t. the parameter s with values in $S^m_\rho(\mathbb{R}^p \times \mathbb{R}^r)$ and define*

$$\tilde{\eta}_s(\Delta, y, \eta) := \eta - \Delta f_s(y, \eta),$$

where f_s is bounded w.r.t. the parameter s with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r, \mathbb{R}^r)$ and homogeneous of degree one in η, for $|\eta| \geq 1$. Then

$$\tilde{p}_s(\Delta, y, \eta) := p_s(y, \tilde{\eta}(\Delta, y, \eta))$$

is bounded w.r.t. s and Δ with values in $S^m_\rho(\mathbb{R}^p \times \mathbb{R}^r)$ for Δ_3 small enough. In the case $\rho = 1$ it is then bounded w.r.t. s with values in $\mathscr{C}^\infty([0, \Delta_3], S^m_\rho(\mathbb{R}^p \times \mathbb{R}^r))$.

Proof. Let Δ_3 be small enough such that $|\eta - \Delta f_s(y, \eta)| \geq C_0 > 0$ if $|\eta| = 1$ and $\Delta \in [0, \Delta_3]$. We then have

$$1 + C_0|\eta| \leq 1 + |\eta - \Delta f_s(y, \eta)| \leq 1 + C_1|\eta|, \ \eta \in \mathbb{R}^r, \ |\eta| \geq 1, \ \Delta \in [0, \Delta_3].$$

This inequality yields the proper estimates for $\partial_y^\alpha \partial_\eta^\beta \tilde{p}_s$ to prove that $\tilde{p}_s \in S^m_\rho(\mathbb{R}^p \times \mathbb{R}^r)$. Bounds w.r.t. to the parameters s and Δ come naturally. In the case $\rho = 1$, derivatives w.r.t. Δ do not affect the symbol order and type. The proof is complete. ■

Following [22] we introduce the following definition.

Definition 2.11. *Let $L \geq 2$. A symbol $q(z, .)$ bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ is said to satisfy Property (P_L) if it is non-negative and satisfies*

$$(P_L) \quad |\partial_y^\alpha \partial_\eta^\beta q(z, y, \eta)| \leq C(1 + |\eta|)^{-|\beta| + (|\alpha| + |\beta|)/L} (1 + q(z, y, \eta))^{1 - (|\alpha| + |\beta|)/L}, \ \ z \in [0, Z], \ y \in \mathbb{R}^p, \ \eta \in \mathbb{R}^r.$$

We then set $\rho = 1 - 1/L$ and $\delta = 1/L$.

Remark 2.12. Suppose $q(z, .)$ as in Definition 2.11 and $|\alpha| + |\beta| \geq L$ then

$$(1 + |\eta|)^{1 - (|\alpha| + |\beta|)/L} \leq C(1 + q(z, y, \eta))^{1 - (|\alpha| + |\beta|)/L}, z \in [0, Z], \ y \in \mathbb{R}^p, \ \eta \in \mathbb{R}^r.$$

Estimate (P_L) is thus clear in this case.

Examples of symbols with such a property with $L > 2$ are given in [22]. In fact we prove that c_1 satisfies Property (P_L) for $L = 2$.

Lemma 2.13. *Let $q(z, y, \eta)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$. If $q \geq 0$ then q satisfies Property (P_L) for $L = 2$.*

Proof. Bounds w.r.t. z are natural; we shall omit the dependence on z in the proof for concision. We have to prove that

$$|\partial_y^\alpha \partial_\eta^\beta q| \leq C\,(1 + |\eta|)^{\frac{1}{2}(|\alpha|-|\beta|)}\,(1 + q)^{1-\frac{1}{2}(|\alpha|+|\beta|)}$$

The property is clearly true for $|\alpha|+|\beta| = 0$ and for $|\alpha|+|\beta| \geq 2$ by the remark above. Let us now treat the case $|\alpha| + |\beta| = 1$. For this we use Landau's inequality: let $f \in \mathscr{C}^2(\mathbb{R})$ with $f \geq 0$ and f'' is bounded then (see [4, page 40] and [10, Lemma 7.7.2])

$$|f'(t)| \leq 2\,(f(t))^{\frac{1}{2}}\,\left(\sup_{t\in\mathbb{R}} |f''(t)|\right)^{\frac{1}{2}}.$$

We first treat the case $|\alpha| = 1$. Define $p(y, \eta) = (1+|\eta|^2)^{-\frac{1}{2}}\,q(y, \eta)$. Then $p \in S^0(\mathbb{R}^p \times \mathbb{R}^r)$ and $\partial_y^{2\alpha} p(y, \eta)$ is in $S^0(\mathbb{R}^p \times \mathbb{R}^r)$ and is thus bounded. We thus have

$$(1 + |\eta|^2)^{-\frac{1}{2}}\,|\partial_y^\alpha q(y, \eta)| \leq C\,((1 + |\eta|^2)^{-\frac{1}{2}}\,q(y, \eta))^{\frac{1}{2}},$$

which yields

$$|\partial_y^\alpha q(y, \eta)| \leq C\,(1 + |\eta|)^{\frac{1}{2}}\,(1 + q(y, \eta))^{\frac{1}{2}},$$

which is the expected estimate. Let us now treat the case $|\beta| = 1$, with for instance, $\beta = (1, 0, \ldots, 0)$ and $\alpha = (0, \ldots, 0)$. Define $p(y, \eta) = (1 + |\eta|^2)^{\frac{1}{2}}\,q(y, \eta)$. Then $p \in S^2(\mathbb{R}^p \times \mathbb{R}^r)$ and thus $\partial_\eta^{2\beta} p(y, \eta)$ is bounded. We hence have

$$|\partial_\eta^\beta p(y, \eta)| \leq C\,(p(y, \eta))^{\frac{1}{2}}.$$

With

$$\partial_\eta^\beta p(y, \eta) = (1 + |\eta|^2)^{\frac{1}{2}}\,\partial_\eta^\beta q(y, \eta) + \eta_1 (1 + |\eta|^2)^{-\frac{1}{2}}\,q(y, \eta),$$

the triangular inequality yields

$$\begin{aligned}(1 + |\eta|^2)^{\frac{1}{2}}\,|\partial_\eta^\beta q(y, \eta)| &\leq C\,(p(y, \eta))^{\frac{1}{2}} + |\eta_1|(1 + |\eta|^2)^{-\frac{1}{2}}\,q(y, \eta)\\ &\leq C\,(q(y, \eta))^{\frac{1}{2}}((1 + |\eta|^2)^{\frac{1}{4}} + (q(y, \eta))^{\frac{1}{2}})\\ &\leq C\,(q(y, \eta))^{\frac{1}{2}}\,(1 + |\eta|^2)^{\frac{1}{4}}.\end{aligned}$$

We finally obtain

$$|\partial_\eta^\beta q(y, \eta)| \leq C\,(q(y, \eta))^{\frac{1}{2}}(1 + |\eta|)^{-\frac{1}{2}},$$

which is the expected estimate. ∎

Remark 2.14. If the symbol $q(z, y, \eta)$ satisfies Property (P_L) then the amplitude $q(z, y', \eta)+ q(z, y, \eta)$ also satisfies Property (P_L) (with derivatives w.r.t. y, y' and η).

Proposition 2.15. *Let $q(z, .)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ with $q(z, .) \geq 0$. Let $q(z, .)$ satisfy Property (P_L) and define $\rho_\Delta(z, y, \eta) = \exp[-\Delta q(z, y, \eta)]$. Let $m \in \mathbb{N}$. Then $q^m \rho_\Delta$ is smooth w.r.t. Δ, bounded w.r.t. z, with values in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^r)$ for Δ in any interval $[\Delta_{min}, \Delta_{max}]$ with $\Delta_{min} > 0$.*

Proof. $\partial_y^\alpha \partial_\eta^\beta (q^m \rho_\Delta)$ is a linear combination of terms of the form

$$\Delta^k (\partial_y^{a_1} \partial_\eta^{b_1} q) \dots (\partial_y^{a_l} \partial_\eta^{b_l} q)(\partial_y^{\alpha_1} \partial_\eta^{\beta_1} q) \dots (\partial_y^{\alpha_k} \partial_\eta^{\beta_k} q) q^{m-l} \rho_\Delta$$

with $0 \le l \le m$ and $a_1 + \cdots + a_l + \alpha_1 + \dots \alpha_k = \alpha$ and $b_1 + \cdots + b_l + \beta_1 + \dots \beta_k = \beta$. We can estimate the absolute value of each of these terms, using Property (P_L), by

$$C\Delta^k (1 + |\eta|)^{-|\beta| + \frac{|\alpha|+|\beta|}{L}} (1 + q)^{l+k-\frac{|\alpha|+|\beta|}{L}} q^{m-l} \rho_\Delta \le C(1 + |\eta|)^{-|\beta| + \frac{|\alpha|+|\beta|}{L}} \Delta_{min}^{-m}$$

as $(1 + q)^{l+k-\frac{|\alpha|+|\beta|}{L}} q^{m-l} \Delta^{k+m} \rho_\Delta \le C$. ∎

While the symbol $\exp[-\Delta q(z, y, \eta)]$ is bounded w.r.t. z and smooth w.r.t. Δ with $\Delta \ge \Delta_{min} > 0$ with values in $S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r)$, this fails to be true at $\Delta = 0$:

$$\partial_\Delta \exp[-\Delta q]|_{\Delta=0} = -q \notin S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r).$$

In fact when we want to control the behavior of $\exp[-\Delta q]$ close to $\Delta = 0$ we shall use the following definition and lemmas.

Definition 2.16. *Let $L \ge 2$, $\rho = 1 - 1/L$ and $\delta = 1/L$. Let $\rho_\Delta(z, y, \eta)$ be a function in $\mathscr{C}^\infty(\mathbb{R}^p \times \mathbb{R}^r)$ depending on the parameters $\Delta \ge 0$ and $z \in [0, Z]$. We say that ρ_Δ satisfies Property (Q_L) if the following holds*

$$(Q_L) \quad \partial_y^\alpha \partial_\eta^\beta (\rho_\Delta - \rho_\Delta|_{\Delta=0})(z, y, \eta) = \Delta^{m+\delta(|\alpha|+|\beta|)} \rho_\Delta^{m\alpha\beta}(z, y, \eta),$$
$$\text{for } |\alpha| + |\beta| \le L, \quad 0 \le m \le 1 - \delta(|\alpha| + |\beta|),$$

where $\rho_\Delta^{m\alpha\beta}(z, y, \eta)$ is bounded w.r.t. Δ and z with values in $S_\rho^{m-\rho|\beta|+\delta|\alpha|}(\mathbb{R}^p \times \mathbb{R}^r)$. It follows that $\rho_\Delta(z, y, \eta) - \rho_\Delta|_{\Delta=0}(z, y, \eta)$ is itself bounded w.r.t. Δ and z with values in $S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r)$.

Lemma 2.17. *Let $q(z, .)$ be bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ and satisfy Property (P_L). Define $\rho_\Delta(z, y, \eta) = \exp[-\Delta q(z, y, \eta)]$. Then ρ_Δ satisfies Property (Q_L) for $\Delta \in [0, \Delta_{max}]$ for any $\Delta_{max} > 0$. As $\rho_\Delta|_{\Delta=0} = 1$, ρ_Δ is itself bounded w.r.t. Δ and z with values in $S_\rho^0(\mathbb{R}^p \times \mathbb{R}^r)$.*

Proof. In the proof all the functions and symbols will naturally be bounded w.r.t. z. We thus drop the variable z here for concision.

We define

$$\rho_\Delta^{m\alpha\beta} := \Delta^{-m-\delta(|\alpha|+|\beta|)} \partial_y^\alpha \partial_\eta^\beta (\rho_\Delta - \rho_\Delta|_{\Delta=0}).$$

We first consider the case $|\alpha| + |\beta| = 0$ with $0 \le m \le 1$. We need to estimate $|\partial_y^a \partial_\eta^b \rho_\Delta^{m00}|$. The case $m = 0$, $|a + b| = 0$ has to be treated independently but is trivial: we clearly have $|\rho_\Delta^{000}| = |\rho_\Delta - 1| \le C$. We shall now estimate $|\partial_y^a \partial_\eta^b \rho_\Delta^{m00}| = |\Delta^{-m} \partial_y^a \partial_\eta^b (\rho_\Delta - 1)|$ in the case where $m > 0$ or $|a| + |b| > 0$. For this we write

$$\rho_\Delta(y, \eta) - 1 = -\Delta \int_0^1 q(y, \eta) \exp[-s\Delta q(y, \eta)] ds. \tag{2.14}$$

We then have $\rho_\Delta^{m00}(y,\eta) = -\int\limits_0^1 q_\Delta^m(s,y,\eta)ds$ with

$$q_\Delta^m(s,y,\eta) = \Delta^{1-m} q(y,\eta)\exp[-s\Delta q(y,\eta)].$$

We prove that

$$|\partial_y^a \partial_\eta^b q_\Delta^m(s,y,\eta)| \le C(s)(1+|\eta|)^{m-\rho|b|+\delta|a|}$$

with $C(s)$ bounded w.r.t. Δ and L^1 w.r.t. $s \in [0,1]$. The result then follows for ρ_Δ^{m00}.

When computing $\partial_y^a \partial_\eta^b q_\Delta^m$ we obtain a linear combination of terms of the form

$$\Delta^{1-m}(\partial_y^{a_0}\partial_\eta^{b_0} q)(-s\Delta)^k(\partial_y^{a_1}\partial_\eta^{b_1} q)\dots(\partial_y^{a_k}\partial_\eta^{b_k} q)\exp[-s\Delta q],$$
$$\text{with } a_0+a_1+\cdots+a_k = a,\quad b_0+b_1+\cdots+b_k = b,$$

(where k can be 0). Using Property (P_L) we find that the absolute value of such a term is bounded by

$$\begin{aligned} &C\Delta^{1-m}(s\Delta)^k(1+|\eta|)^{-|b|+\delta(|a+b|)}(1+q)^{k+1-\delta(|a+b|)}\exp[-s\Delta q] \\ &\quad \le C s^{m+\delta(|a+b|)-1}(1+|\eta|)^{m-\rho|b|+\delta|a|}\Delta^{\delta(|a+b|)}(s\Delta(1+q))^{-m+k+1-\delta(|a+b|)}\exp[-s\Delta q], \end{aligned}$$

as $1 \le C(1+|\eta|)^m(1+q)^{-m}$ if $m \ge 0$. If $l := -m+k+1-\delta(|a+b|) \ge 0$ we use that $(s\Delta(1+q))^l\exp[-s\Delta q] \le C$ if $0 \le s \le 1$, $0 \le \Delta \le \Delta_{max}$ and $q \ge 0$ and we obtain the following estimate

$$C s^{m+\delta(|a+b|)-1}(1+|\eta|)^{m-\rho|b|+\delta|a|}\Delta^{\delta(|a+b|)}.$$

If $l < 0$, $(1+q)^l$ is simply bounded ($q \ge 0$) and we obtain the following estimate:

$$C\Delta^{k+1-m}s^k(1+|\eta|)^{m-\rho|b|+\delta|a|}.$$

As $m+\delta(|a+b|)-1 > -1$ in the considered case, both estimates exhibit bounds that are in $L^1([0,1])$ w.r.t. s. We also have uniform bounds w.r.t. Δ as we have assumed $m \le 1$.

We now treat the case $1 \le |\alpha|+|\beta| \le L$, $0 \le m \le 1-\delta(|\alpha|+|\beta|)$. We estimate the absolute value of

$$\partial_y^a\partial_\eta^b(\rho_\Delta^{m\alpha\beta}) = \Delta^{-m-\delta(|\alpha|+|\beta|)}\ \partial_y^{a+\alpha}\partial_\eta^{b+\beta}\rho_\Delta,$$

which is a linear combination of terms of the form

$$\Delta^{k-m-\delta(|\alpha|+|\beta|)}\ (\partial_y^{a_1}\partial_\eta^{b_1} q)\dots(\partial_y^{a_k}\partial_\eta^{b_k} q)\exp[-\Delta q],$$

with $a_1+\cdots+a_k = a+\alpha$, $b_1+\cdots+b_k = b+\beta$, where $k \ge 1$. Using Property (P_L) we find that the absolute value of such a term is bounded by

$$\begin{aligned} &C\Delta^{k-m-\delta(|\alpha|+|\beta|)}(1+|\eta|)^{-|\beta|-|b|+\delta(|\alpha|+|a|+|\beta|+|b|)}(1+q)^{k-\delta(|\alpha|+|a|+|\beta|+|b|)}\exp[-\Delta q] \\ &\quad \le C(1+|\eta|)^{m-\rho(|\beta|+|b|)+\delta(|\alpha|+|a|)}(1+q)^{-\delta(|a|+|b|)}(\Delta(1+q))^{k-m-\delta(|\alpha|+|\beta|)}\exp[-\Delta q] \\ &\quad \le C(1+|\eta|)^{m-\rho(|\beta|+|b|)+\delta(|\alpha|+|a|)}, \end{aligned}$$

as $k-m-\delta(|\alpha|+|\beta|) \ge 1-m-\delta(|\alpha|+|\beta|) \ge 0$ and $0 \le \Delta \le \Delta_{max}$. This completes the proof. ∎

Lemma 2.18. *Let $p_\Delta(z,y,\eta) \in S^0_\rho(\mathbb{R}^p\times\mathbb{R}^r)$ satisfy Property (Q_L), such that $p_\Delta|_{\Delta=0}$ is constant. Let $f_\Delta(z,y,\eta)$ be bounded w.r.t. z and Δ with values in $S^1(\mathbb{R}^p\times\mathbb{R}^r)$ be homogeneous of degree one in η for $|\eta|\geq 1$. Define $\tilde\eta(\Delta,z,y,\eta) := \eta - \Delta f_\Delta(z,y,\eta)$. Then*

$$\tilde p_\Delta(z,y,\eta) := p_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta))$$

satisfies property (Q_L) for Δ sufficiently small.

Proof. Take Δ small enough such that Lemma 2.10 applies. We first treat the case $\alpha = 0, \beta = 0$. Let $h = p_\Delta|_{\Delta=0}$. Property (Q_L) gives

$$\begin{aligned}\tilde p_\Delta(z,y,\eta) - \tilde p_\Delta(z,y,\eta)|_{\Delta=0} &= p_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta)) - h\\ &= \Delta^m q^{m00}_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta)),\quad 0\leq m\leq 1\end{aligned}$$

with $q^{m00}_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta))$ bounded w.r.t. z and Δ with values in $S^m_\rho(\mathbb{R}^p\times\mathbb{R}^r)$. Let now $1\leq|\alpha|+|\beta|\leq L$. $\partial^\alpha_y\partial^\beta_\eta\tilde p_\Delta(z,y,\eta)$ is a linear combination of terms of the form

$$\partial^{\alpha_0}_y\partial^{\beta_0}_\eta p_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta))\,\partial^{\alpha_1}_y\partial^{\beta_1}_\eta\tilde\eta(\Delta,z,y,\eta)\ldots\partial^{\alpha_k}_y\partial^{\beta_k}_\eta\tilde\eta(\Delta,z,y,\eta)$$

with $|\beta_0| = k, \beta = \beta_1+\cdots+\beta_k$ and $\alpha = \alpha_0+\cdots+\alpha_k$. Note that $k\geq 1$ and $|\alpha_i|+|\beta_i|\geq 1$, $i = 1,\ldots k$. By Property (Q_L) this term is thus of the form

$$\Delta^{m+\delta(|\alpha_0|+|\beta_0|)}q^{m\alpha_0\beta_0}_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta))\,\partial^{\alpha_1}_y\partial^{\beta_1}_\eta\tilde\eta(\Delta,z,y,\eta)\ldots\partial^{\alpha_k}_y\partial^{\beta_k}_\eta\tilde\eta(\Delta,z,y,\eta), \tag{2.15}$$

with $0\leq m\leq 1-\delta(|\alpha_0|+|\beta_0|)$ and $q^{m\alpha_0\beta_0}_\Delta(z,y,\tilde\eta(\Delta,z,y,\eta))$ bounded w.r.t. z and Δ with values in $S^{m-\rho|\beta_0|+\delta|\alpha_0|}_\rho(\mathbb{R}^p\times\mathbb{R}^r)$ by Lemma 2.10 .

Assume first that, for this term, $|\alpha_i|+|\beta_i| = 1$ for all $i = 1,\ldots,k$. Then $\sum_{i=1}^k|\alpha_i|+|\beta_i| = k$ and $|\alpha_0|+|\beta_0| = |\alpha|+|\beta|$. The term $\partial^{\alpha_i}_y\partial^{\beta_i}_\eta\tilde\eta(\Delta,z,y,\eta)$ in the product (2.15) belongs to $S^{1-\beta_i}(\mathbb{R}^p\times\mathbb{R}^r)$ and thus (2.15) is of the form $\Delta^{m+\delta(|\alpha|+|\beta|)}\tilde q^{m\alpha_0\beta_0}_\Delta(z,y,\eta)$ with $\tilde q^{m\alpha_0\beta_0}_\Delta$ in $S^l_\rho(\mathbb{R}^p\times\mathbb{R}^r)$ with $l = m-\rho|\beta_0|+\delta|\alpha_0|+k-|\beta_1|-\cdots-|\beta_k| = m+\delta(|\beta_0|+|\alpha_0|)-|\beta|$. As $|\alpha_0|+|\beta_0| = |\alpha|+|\beta|$ we have $l = m+\delta(|\alpha|+|\beta|)-|\beta| = m-\rho|\beta|+\delta|\alpha|$. We thus obtain the expected result in this case.

Assume now that there exists $i\in\{1,\ldots,k\}$ such that $|\alpha_i|+|\beta_i|\geq 2$. Then the term $\partial^{\alpha_i}_y\partial^{\beta_i}_\eta\tilde\eta(\Delta,z,y,\eta)$ in the product (2.15) is equal to $\Delta\partial^{\alpha_i}_y\partial^{\beta_i}_\eta f_\Delta(z,y,\eta)$. Thus the term (2.15) is of the form $\Delta^{1+m+\delta(|\alpha_0|+|\beta_0|)}\tilde{\tilde q}^{m\alpha_0\beta_0}_\Delta(z,y,\eta)$ with $\tilde{\tilde q}^{m\alpha_0\beta_0}_\Delta$ in $S^l_\rho(\mathbb{R}^p\times\mathbb{R}^r)$. As above $l = m+\delta(|\alpha_0|+|\beta_0|)-|\beta|$. In the present case $|\alpha_0|+|\beta_0| < |\alpha|+|\beta|$ which yields $l < m-\rho|\beta|+\delta|\alpha|$ and hence the expected result since $1+m+\delta(|\alpha_0|+|\beta_0|)\geq 1\geq m+\delta(|\alpha|+|\beta|)$. ∎

Lemma 2.19. *Let $f\in\mathscr{C}^\infty(\mathbb{R})$ and $q_\Delta(z,y,\eta)$ in $\mathscr{C}^\infty(\mathbb{R}^p\times\mathbb{R}^r)$ that satisfies Property (Q_L) and such that $q_\Delta(z,.)|_{\Delta=0}$ is independent of y and η. Then $f(q_\Delta)(z,y,\eta)$ satisfies Property (Q_L).*

Proof. Again bounds w.r.t. z are clear. We first treat the case $|\alpha|+|\beta| = 0$. We write

$$f(q_\Delta) - f(q_\Delta|_{\Delta=0}) = (q_\Delta - q_\Delta|_{\Delta=0})\int_0^1 f'((1-s)q_\Delta|_{\Delta=0} + sq_\Delta)ds.$$

As $q_\Delta|_{\Delta=0}$ is independent of y and η, then q_Δ is bounded w.r.t. Δ with values in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^r)$ by Property (Q_L) and so are $(1-s)q_\Delta|_{\Delta=0}+sq_\Delta$ and $f'((1-s)q_\Delta|_{\Delta=0}+sq_\Delta)$ by Lemma 18.1.10 in [8] with bounds in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^r)$ uniform with respect to s. We thus obtain that $\int_0^1 f'((1-s)q_\Delta|_{\Delta=0}+sq_\Delta)ds$ is bounded w.r.t. Δ with values in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^r)$. We conclude using Property (Q_L) for $q_\Delta - q_\Delta|_{\Delta=0}$. Let us now treat the case $1 \le |\alpha|+|\beta| \le L$ and choose $0 \le m \le 1-\delta(|\alpha|+|\beta|)$. We see that $\partial_y^\alpha \partial_\eta^\beta f(q_\Delta)$ is a linear combination of terms of the form

$$(\partial_y^{\alpha_1}\partial_\eta^{\beta_1} q_\Delta)\dots(\partial_y^{\alpha_k}\partial_\eta^{\beta_k} q_\Delta) f^{(k)}(q_\Delta),$$

where $k \ge 1$, $\alpha_1+\dots+\alpha_k = \alpha$, $\beta_1+\dots+\beta_k=\beta$. Now choose $0 \le m_i \le 1-\delta(|\alpha_i|+|\beta_i|)$, $i=1,\dots,k$, such that $m = m_1+\dots+m_k$. Then Property (Q_L) yields terms of the form

$$\Delta^{m_1+\delta(|\alpha_1|+|\beta_1|)}\dots\Delta^{m_k+\delta(|\alpha_k|+|\beta_k|)} q_\Delta^{m_1\alpha_1\beta_1}\dots q_\Delta^{m_k\alpha_k\beta_k} = \Delta^{m+\delta(|\alpha|+|\beta|)} q_\Delta^{m\alpha\beta}$$

with $q_\Delta^{m_i\alpha_i\beta_i}$, $i=1,\dots,k$, bounded w.r.t. Δ with values in $S_\rho^{m_i-\rho|\alpha_i|+\delta|\beta_i|}(\mathbb{R}^p \times \mathbb{R}^r)$ and $q_\Delta^{m\alpha\beta} := q_\Delta^{m_1\alpha_1\beta_1}\dots q_\Delta^{m_k\alpha_k\beta_k}$. We note that $f^{(k)}(q_\Delta)$ is bounded w.r.t. Δ with values in $S^0_\rho(\mathbb{R}^p \times \mathbb{R}^r)$. The symbol $q_\Delta^{m\alpha\beta}$ is bounded w.r.t. Δ with values in $S_\rho^{m-\rho|\alpha|+\delta|\beta|}(\mathbb{R}^p \times \mathbb{R}^r)$, which yields the result. ∎

With Remark 2.14, Lemma 2.19 and the previous lemma we obtain the following corollary.

Corollary 2.20. *Let $f \in \mathscr{C}^\infty(\mathbb{R})$ and let $q(z,.)$ bounded w.r.t. z with values in $S^1(\mathbb{R}^p \times \mathbb{R}^r)$ satisfy Property (P_L). Define*

$$p_\Delta(z,y',y,\eta) = \exp[-\Delta(q(z,y',\eta)+q(z,y,\eta))].$$

Then $f(p_\Delta)$ satisfies Property (Q_L). As $f(p_\Delta)|_{\Delta=0} = f(1)$, $f(p_\Delta)$ is itself bounded w.r.t. Δ and z with values in $S^0_\rho(\mathbb{R}^{2p} \times \mathbb{R}^r)$.

Note that the property (Q_L) is stable when we go from amplitudes to symbols.

Proposition 2.21. *Let $q_\Delta(z,x,y,\xi)$ be an amplitude in $S^0_\rho(\mathbb{R}^{2p} \times \mathbb{R}^p)$ depending on the parameters $\Delta \ge 0$ and $z \in [0,Z]$ that satisfies Property (Q_L). Then $\sigma\{q_\Delta\}(z,x,\xi)$ satisfies property (Q_L).*

Proof. We use the oscillatory integral representation for the symbol

$$\sigma\{q_\Delta\}(z,x,\xi) := \iint \exp[-i\langle y|\eta\rangle]\, q_\Delta(z,x,x-y,\xi-\eta)\, đ\eta\, dy.$$

Let $0 \le |\alpha|+|\beta| \le L$ and $0 \le m \le 1-\delta(|\alpha|+|\beta|)$. Computing $\partial_x^\alpha \partial_\xi^\beta(\sigma\{q_\Delta\} - \sigma\{q_\Delta\}|_{\Delta=0})$, we obtain a linear combination of terms of the form, with $\alpha_1+\alpha_2=\alpha$,

$$\begin{aligned}\iint \exp[-i\langle y|\eta\rangle]\, &\partial_2^{\alpha_1}\partial_3^{\alpha_2}\partial_4^\beta(q_\Delta - q_\Delta|_{\Delta=0})(z,x,x-y,\xi-\eta)\, đ\eta\, dy \\ &= \iint \exp[-i\langle y|\eta\rangle]\, \Delta^{m+\delta(|\alpha|+|\beta|)} q_\Delta^{m(\alpha_1,\alpha_2)\beta}(z,x,x-y,\xi-\eta)\, đ\eta\, dy \\ &= \Delta^{m+\delta(|\alpha|+|\beta|)} \sigma\left\{q_\Delta^{m(\alpha_1,\alpha_2)\beta}\right\},\end{aligned}$$

where $q_\Delta^{m(\alpha_1,\alpha_2)\beta}$ is bounded w.r.t. Δ and z with values in $S_\rho^{m-\rho|\beta|+\delta|\alpha|}(\mathbb{R}^{2p} \times \mathbb{R}^p)$. As the map $a \mapsto \sigma\{a\}$ maps bounded sets into bounded sets the result follows. ∎

We shall also need the following lemma.

Lemma 2.22. *Let $q_\Delta(z,x,y,\xi)$ be an amplitude in $S^0_\rho(\mathbb{R}^{2p}\times\mathbb{R}^p)$ depending on the parameters $\Delta\geq 0$ and $z\in[0,Z]$ that satisfies Property (Q_L) for $1\leq|\alpha|+|\beta|\leq 2$ and such that $q_\Delta(z,.)|_{\Delta=0}$ is independent of (x,y,ξ). Let $r(x,\xi)\in S^s(\mathbb{R}^p\times\mathbb{R}^p)$ for some $s\in\mathbb{R}$. Then*

$$\sigma\{q_\Delta\, r\}(z,x,\xi) - q_\Delta(z,x,x,\xi)\, r(x,\xi) = \Delta^{m+2\delta}\lambda^m_\Delta(z,x,\xi),\ 0\leq m\leq\rho-\delta,$$

where the function $\lambda^m_\Delta(z,x,\xi)$ is bounded w.r.t. Δ and z with values in $S^{m+s-(\rho-\delta)}_\rho(\mathbb{R}^p\times\mathbb{R}^p)$.

Proof. For the sake of concision we take $p=1$ in the proof but it naturally extends to $p\geq 1$. We write $\lambda_\Delta = q_\Delta r$. Using the oscillatory integral representation of $\sigma\{q_\Delta\}$ we obtain

$$\begin{aligned}\sigma\{q_\Delta r\}(z,x,\xi) &- q_\Delta(z,x,x,\xi)r(x,\xi)\\ &= \iint \exp[-i\langle y|\xi-\eta\rangle](\lambda_\Delta(z,x,x-y,\eta)-\lambda_\Delta(z,x,x,\eta)\, đ\eta\, dy.\end{aligned}$$

Taylor's formula yields

$$\begin{aligned}\sigma\{q_\Delta r\}(z,x,\xi) &- q_\Delta(z,x,x,\xi)r(x,\xi)\\ &= \int_0^1\iint -y\exp[-i\langle y|\xi-\eta\rangle]\,\partial_3\lambda_\Delta(z,x,x-sy,\eta)\, đ\eta\, dy\, ds.\end{aligned}$$

With an integration by parts we obtain

$$\begin{aligned}\sigma\{q_\Delta r\}(z,x,\xi) &- q_\Delta(z,x,x,\xi)r(x,\xi)\\ &= -\int_0^1\iint i\exp[-i\langle y|\xi-\eta\rangle]\,\partial_3\partial_4\lambda_\Delta(z,x,x-sy,\eta)\, đ\eta\, dy\, ds\\ &= \sigma\left\{-i\int_0^1 \partial_3\partial_4\lambda_\Delta(z,x,(1-s)x+sy,\xi)\, ds\right\},\end{aligned}$$

where $\partial_3\partial_4\lambda_\Delta(z,x,y,\xi) = (\partial_y\partial_\xi q_\Delta)(z,x,y,\xi)\, r(x,\xi) + \partial_y q_\Delta(z,x,y,\xi)\partial_\xi r(x,\xi)$, as r does not depend on y. The first term is treated using Property (Q_L) while for the second one we write

$$\partial_y q_\Delta\, \partial_\xi r = \Delta^{m'+\delta}\, q^{m'(0,1)0}_\Delta\, \partial_\xi r,$$

where $0\leq m'\leq 1-\delta$ and $q^{m'(0,1)0}_\Delta\, r\in S^{m'+s-1+\delta}_\rho(\mathbb{R}^{2p}\times\mathbb{R}^p)$ by Property (Q_L). We actually take $\delta\leq m'\leq 1-\delta$ and write $m=m'-\delta$. We obtain

$$\partial_y q_\Delta\partial_\xi r = \Delta^{m+2\delta}\,\tilde{q}^m_\Delta,$$

where $\tilde{q}^m_\Delta$ is bounded w.r.t. Δ with values in $S^{m+s-\rho+\delta}_\rho(\mathbb{R}^{2p}\times\mathbb{R}^p)$ and $0\leq m\leq 1-2\delta=\rho-\delta$. We conclude since the map $\sigma\{.\}$ maps bounded sets into bounded sets. ∎

We shall need the following result.

Proposition 2.23. *Let $\frac{1}{2} \leq \rho \leq 1$ and set $\delta = 1-\rho$. Let $p(x,\xi)$ be a real non-negative $\mathscr{C}^\infty$ function that satisfies*

$$\|p(x,\xi)\| \leq C\langle\xi\rangle, \tag{2.16}$$

$$\|\partial_x^\alpha p(x,\xi)\| \leq C_\alpha\langle\xi\rangle, \quad |\alpha| = 1, \quad \|\partial_\xi^\beta p(x,\xi)\| \leq C_\beta, \quad |\beta| = 1, \tag{2.17}$$

and

$$\partial_x^\alpha \partial_\xi^\beta p(x,\xi) \in S_\rho^{\rho-\delta+\delta|\alpha|-\rho|\beta|}(X\times\mathbb{R}^n), \quad \text{for } |\alpha+\beta| = 2. \tag{2.18}$$

Then there exists a non-negative constant C such that

$$Re(p(x,D)u,u)_{(L^2,L^2)} \geq -C\|u\|_{L^2}^2, \quad u \in \mathscr{S}(\mathbb{R}^n).$$

The constant C can be chosen uniformly if the symbol p remains in a set such that the constants in (2.16)–(2.17) are uniform and if $\partial_x^\alpha\partial_\xi^\beta p(x,\xi)$, $|\alpha+\beta| = 2$, remain in bounded domains of $S_\rho^{\rho-\delta+\delta|\alpha|-\rho|\beta|}(X\times\mathbb{R}^n)$ respectively.

In other words, for the partial differentiation of order zero and one the symbol p behaves like an element $S^1_{1,0}$ and like an element of $S_\rho^{\rho-\delta}$ for higher-order derivatives. The result we prove is of the form of the sharp Gårding inequality. Note that considering $p(x,\xi)$ to be in $S^1_\rho(X\times\mathbb{R}^n)$, we cannot directly apply the usual sharp Gårding inequality to obtain a lower L^2 bound when $\frac{1}{2} \leq \rho < 1$.

Proof. We follow the proof of the sharp Gårding inequality as given in [14, Section 3.4] and [25, Chapter VII]. We introduce the following function

$$F(\xi,\zeta) = \langle\xi\rangle^{-n/4} q(\langle\xi\rangle^{-\frac{1}{2}}(\zeta-\xi)), \tag{2.19}$$

where q is real, even, belongs to $\mathscr{C}^\infty(|\xi| \leq 1)$ and satisfies $\int q(\xi)^2\, d\xi = 1$ and we set

$$\mu(\zeta,x,\xi) = \int F(\zeta,\eta)p(x,\eta)F(\xi,\eta)\, d\eta,$$

which is the so-called Friedrichs' symmetrization of the symbol p. By Theorem 3.4.2 in [14], since $p(x,\xi) \in S^1_\rho(X\times\mathbb{R}^n)$, the function $\mu(\zeta,x,\xi)$ is a double symbol that belongs to $S_\rho^{1,0}(\mathbb{R}^n\times X\times\mathbb{R}^n)$. For the notion of double symbols see [14]. Note that since we are not interested in an asymptotic formula for $\mu(\zeta,y,\xi)$ the first part of the proof of Theorem 3.4.2 in [14] applies to the case $\rho = \delta = \frac{1}{2}$. Then we have $\mu(D_x,y,D_y) = \nu(x,D_x)$, with the symbol $\nu(x,\xi) \in S^1_\rho(X\times\mathbb{R}^n)$ given by [14, Theorem 2.2.5]

$$\nu(x,\xi) = \iint \exp[-i\langle y|\xi-\zeta\rangle]\mu(\zeta,x-y,\xi)\, dy\, đ\zeta,$$

as an oscillatory integral. The operator $\nu(x,D_x)$ is formally self-adjoint and $\nu(x,D_x)$ is non-negative as an operator, i.e., for $u \in \mathscr{S}(\mathbb{R}^n)$ we have [14, Theorem 3.4.3]

$$(\nu(x,D_x)u,u)_{(L^2,L^2)} \geq 0.$$

Next, we study the properties of $(\nu - p)(x,\xi)$. We set

$$\begin{aligned}\nu_0(x,\xi) = \mu(\xi,x,\xi) &= \int F(\xi,\eta)^2 p(x,\eta)d\eta = \int p(x,\xi+\sigma\langle\xi\rangle^{\frac{1}{2}})q(\sigma)^2\, d\sigma \\ &= p(x,\xi) + \int\int_0^1 (1-s)\partial_\xi^2 p(x,\xi+s\sigma\langle\xi\rangle^{\frac{1}{2}};\sigma\langle\xi\rangle^{\frac{1}{2}},\sigma\langle\xi\rangle^{\frac{1}{2}})\, ds\, q(\sigma)^2\, d\sigma,\end{aligned}$$

by the second-order Taylor formula using that $\int q(\sigma)^2 d\sigma = 1$ and that q is even. We observe that $|\sigma| \le 1$ on the support of the function $q(\sigma)$ which gives

$$C\langle\xi\rangle \le \langle\xi + \langle\xi\rangle^{\frac{1}{2}}\sigma\rangle \le C'\langle\xi\rangle. \tag{2.20}$$

From (2.18) we thus obtain that $\nu_0(x,\xi) - p(x,\xi) \in S^0_\rho(X\times\mathbb{R}^n)$. We now prove the following lemma.

Lemma 2.24. *The symbol $(\nu - \nu_0)(x,\xi)$ belongs to $S^0_\rho(X\times\mathbb{R}^n)$.*

Proof. We first define

$$\nu_1(x,\xi) = -i\sum_j \partial_{x_j}\partial_{\zeta_j}\mu(\zeta,x,\xi)|_{\zeta=\xi} = -i\sum_j \int \partial_{\xi_j}F(\xi,\eta)\,\partial_{x_j}p(x,\eta)\,F(\xi,\eta)\,d\eta \tag{2.21}$$

and prove that $\nu_1(x,\xi)\in S^0_\rho(X\times\mathbb{R}^n)$. From [14, Lemma 3.4.1] $\partial_{\xi_j}F(\xi,\eta)$ has the form

$$\partial_{\xi_j}F(\xi,\eta) = \langle\xi\rangle^{-n/4}\sum_{|\gamma|\le 1,\ \gamma_1\le\gamma}\psi_{\gamma,\gamma_1}(\xi)\,\left((\eta-\xi)\langle\xi\rangle^{-\frac{1}{2}}\right)^{\gamma_1}(\partial_\sigma^\gamma q)((\eta-\xi)\langle\xi\rangle^{-\frac{1}{2}}),$$

where $\psi_{\gamma,\gamma_1}\in S^{-1+\frac{1}{2}|\gamma-\gamma_1|}_{1,0}(\mathbb{R}^n)$. From the definition of F in (2.19) we write the jth term in the sum in (2.21) as

$$\begin{aligned}\nu_1^{(j)}(x,\xi) &= -i\langle\xi\rangle^{-n/2}\sum_{|\gamma|\le 1,\ \gamma_1\le\gamma}\psi_{\gamma,\gamma_1}(\xi)\int\left((\eta-\xi)\langle\xi\rangle^{-\frac{1}{2}}\right)^{\gamma_1}(\partial_\sigma^\gamma q)((\eta-\xi)\langle\xi\rangle^{-\frac{1}{2}}) \\ &\qquad\times\partial_{x_j}p(x,\eta)\,q((\eta-\xi)\langle\xi\rangle^{-\frac{1}{2}})\,d\eta \\ &= -i\sum_{|\gamma|\le 1,\ \gamma_1\le\gamma}\psi_{\gamma,\gamma_1}(\xi)\int\sigma^{\gamma_1}(\partial_\sigma^\gamma q)(\sigma)\,q(\sigma)\,\partial_{x_j}p(x,\xi+\langle\xi\rangle^{\frac{1}{2}}\sigma)\,d\sigma,\end{aligned} \tag{2.22}$$

after a change of variable. There are two cases to consider in the sum in (2.22): a) $\gamma_1 = \gamma$ and b) $\gamma_1 = 0$ and $|\gamma| = 1$. From (2.20), and (2.17) and from the fact that $\psi_{\gamma,\gamma}\in S^{-1}_{1,0}(\mathbb{R}^n)$ we find that the contribution from case *a*), i.e.,

$$\nu_1^{(j,a)}(x,\xi) = -i\sum_{|\gamma|\le 1}\psi_{\gamma,\gamma}(\xi)\int\sigma^\gamma(\partial_\sigma^\gamma q)(\sigma)\,q(\sigma)\,\partial_{x_j}p(x,\xi+\langle\xi\rangle^{\frac{1}{2}}\sigma)\,d\sigma,$$

satisfies $|\nu_1^{(j,a)}(x,\xi)|\le C$. Computing $\partial_x^\alpha\partial_\xi^\beta\nu_1^{(j,a)}(x,\xi)$ we find it to be a linear combination of terms of the form

$$\sum_{|\gamma|\le 1}\partial_\xi^{\beta_1}\psi_{\gamma,\gamma}(\xi)\int\sigma^\gamma(\partial_\sigma^\gamma q)(\sigma)\,q(\sigma)\,\partial_x^\alpha\partial_\xi^{\beta_2}\partial_{x_j}p(x,\xi+\langle\xi\rangle^{\frac{1}{2}}\sigma)\,d\sigma,\quad \text{with } \beta_1+\beta_2=\beta.$$

From (2.20) and (2.18) we see that $\partial_x^\alpha \partial_\xi^{\beta_2} \partial_{x_j} p(x, \xi + \langle\xi\rangle^{\frac{1}{2}}\sigma)$ is in $S_\rho^{\delta|\alpha|+\rho(1-|\beta_2|)}(X \times \mathbb{R}^n)$ uniformly w.r.t. σ, $|\sigma| \leq 1$ and $\partial_\xi^{\beta_1} \psi_{\gamma,\gamma}(\xi)$ is in $S_{1,0}^{-1-|\beta_1|}(X \times \mathbb{R}^n)$. As a result, $v_1^{(j,a)}(x, \xi)$ belongs to $S_\rho^0(X \times \mathbb{R}^n)$. We now consider the contribution from case b) in the sum in (2.22), i.e.,

$$v_1^{(j,b)}(x, \xi) = -i \sum_{|\gamma|=1} \psi_{\gamma,0}(\xi) \int (\partial_\sigma^\gamma q)(\sigma)\, q(\sigma)\, \partial_{x_j} p(x, \xi + \langle\xi\rangle^{\frac{1}{2}}\sigma)\, d\sigma.$$

We write

$$\partial_{x_j} p(x, \xi + \langle\xi\rangle^{\frac{1}{2}}\sigma) = \partial_{x_j} p(x, \xi) + \langle\xi\rangle^{\frac{1}{2}} \sum_l \sigma_l \int_0^1 (\partial_{\xi_l} \partial_{x_j} p)(x, \xi + t\sigma\langle\xi\rangle^{\frac{1}{2}})\, dt.$$

Since $\int \partial_\sigma^\gamma q(\sigma)\, q(\sigma)\, d\sigma = 0$ (q is even), the first term gives no contribution and we obtain

$$v_1^{(j,b)}(x, \xi) = -i\langle\xi\rangle^{\frac{1}{2}} \sum_{|\gamma|=1,l} \psi_{\gamma,0}(\xi) \int \int_0^1 (\partial_\sigma^\gamma q)(\sigma)\, q(\sigma)\, \sigma_l (\partial_{\xi_l} \partial_{x_j} p)(x, \xi + t\sigma\langle\xi\rangle^{\frac{1}{2}})\, dt\, d\sigma.$$

Since by (2.18), $\partial_{\xi_l} \partial_{x_j} p(x, \xi) \in S_\rho^0(X \times \mathbb{R}^n)$ and $\psi_{\gamma,0}(\xi) \in S_{1,0}^{-\frac{1}{2}}(\mathbb{R}^n)$ we obtain that $v_1^{(j,b)}(x, \xi) \in S_\rho^0(X \times \mathbb{R}^n)$ from (2.20).

We have thus proven that $v_1(x, \xi) \in S_\rho^0(X \times \mathbb{R}^n)$. We now compute $(v - v_0)(x, \xi)$.

$$\begin{aligned}
(v - v_0)(x, \xi) = &\iiint \exp[-i\langle y|\xi - \zeta\rangle] F(\zeta, \eta) p(x - y, \eta) F(\xi, \eta)\, dy\, đ\zeta\, d\eta \\
&\qquad - \int F(\xi, \eta)^2 p(x, \eta)\, d\eta \\
&= \iiint \exp[-i\langle y|\xi - \zeta\rangle] F(\zeta, \eta)\, (p(x - y, \eta) - p(x, \eta))\, F(\xi, \eta)\, dy\, đ\zeta\, d\eta \\
&= \int_0^1 \sum_j \iiint -y_j \exp[-i\langle y|\xi - \zeta\rangle] F(\zeta, \eta) \partial_{x_j} p(x - sy, \eta) F(\xi, \eta)\, dy\, đ\zeta\, d\eta\, ds \\
&= -i \int_0^1 \sum_j \iiint \exp[-i\langle y|\xi - \zeta\rangle] \partial_{\zeta_j} F(\zeta, \eta) \partial_{x_j} p(x - sy, \eta) F(\xi, \eta)\, dy\, đ\zeta\, d\eta\, ds,
\end{aligned}$$

after an integration by parts. Arguing similarly, computing $(v - v_0 - v_1)(x, \xi)$, we obtain

$$\begin{aligned}
(v - v_0 - v_1)(x, \xi) = -i \int_0^1 \sum_j \iiint &\exp[-i\langle y|\xi - \zeta\rangle] \partial_{\zeta_j} F(\zeta, \eta) \\
&\times \left(\partial_{x_j} p(x - sy, \eta) - \partial_{x_j} p(x, \eta)\right) F(\xi, \eta)\, dy\, đ\zeta\, d\eta\, ds \\
= - \int_0^1 \int_0^1 s \sum_{j,l} \iiint &\exp[-i\langle y|\xi - \zeta\rangle] \partial^2_{\zeta_j, \zeta_l} F(\zeta, \eta) \\
&\times \partial^2_{x_j, x_l} p(x - s' sy, \eta) F(\xi, \eta)\, dy\, đ\zeta\, d\eta\, ds\, ds' \\
= -\sigma \left\{ \int_0^1 \int_0^1 s \sum_{j,l} \int \right. &\left. \partial^2_{\zeta_j, \zeta_l} F(\zeta, \eta) \partial^2_{x_j, x_l} p((1 - s's)x + s'sy, \eta) F(\xi, \eta) d\eta ds\, ds' \right\}.
\end{aligned}$$

Observing that

$$\tilde{p}(x,y,\eta) := \int_0^1\int_0^1 s\partial^2_{x_j,x_l} p((1-s's)x + ss'y,\eta)\, ds\, ds'$$

is in $S^1_\rho(X\times Y\times\mathbb{R}^n)$ by (2.18), and then following the proof of Theorem 3.4.2 in [14] we find that its Friedrichs' symmetrization,

$$\tilde{\mu}(x,\zeta,y,\xi) = \int F(\zeta,\eta)\tilde{p}(x,y,\eta)F(\xi,\eta)\, d\eta,$$

is in $S^{1,0}_\rho(X\times\mathbb{R}^n\times Y\times\mathbb{R}^n)$ and thus $\partial^2_{\zeta_j,\zeta_l}\tilde{\mu}(x,\zeta,y,\xi)$ is in $S^{0,0}_\rho(X\times\mathbb{R}^n\times Y\times\mathbb{R}^n)$ and finally we find $(\nu-\nu_0-\nu_1)(x,\xi)\in S^0_\rho(X\times\mathbb{R}^n)$ by Theorem 2.2.5 in [14]. With $\nu_1(x,\xi)\in S^0_\rho(X\times\mathbb{R}^n)$ as proven above, this concludes the proof. ∎

End of the proof of Proposition 2.23. As a consequence of the previous lemma we find that $(\nu-p)(x,\xi)\in S^0_\rho(X\times\mathbb{R}^n)$ and we have

$$\begin{aligned}\mathrm{Re}(p(x,D_x)u,u)_{(L^2,L^2)} &= (\nu(x,D_x)u,u)_{(L^2,L^2)} + \mathrm{Re}((p-\nu)(x,D_x)u,u)_{(L^2,L^2)}\\ &\geq -C\|u\|^2_{L^2},\end{aligned}$$

by the Calderón-Vaillancourt Theorem (see [14, Chapter 7, Sections 1,2] or [25, Section XIII-2]). ∎

The following result is at the heart of the precise Sobolev operator-norm estimation of the thin-slab propagator $\mathcal{G}_{(z',z)}$.

Theorem 2.25. *Let* $\rho_\Delta(z,x,\xi) = \exp[-\Delta q(z,x,\xi)]$ *with* $q(z,x,\xi)$ *satisfying Property* (P_L)*. There exist* $\Delta_4 > 0$ *and* $C\geq 0$ *such that*

$$\|\rho_\Delta(z,x,D_x)\|_{(L^2,L^2)} \leq 1 + C\Delta,$$

for all $z', z\in[0,Z]$ *such that* $0\leq\Delta = z'-z\leq\Delta_4$.

Proof. In the proof, we shall always assume that Δ is sufficiently small to apply the invoked properties and results. By Lemma 2.17 $\rho_\Delta(z,x,\xi)$ satisfies Property (Q_L). We prove that $(\rho_\Delta(z,x,D_x)\circ\rho_\Delta(z,x,D_x)^*u,u)_{(L^2,L^2)} \leq (1+C\Delta)\|u\|^2_{L^2}$ for all $u\in\mathscr{S}(\mathbb{R}^n)$. The ψDO $\rho_\Delta(z,x,D_x)\circ\rho_\Delta(z,x,D_x)^*$ admits the amplitude

$$p_\Delta(z,x,y,\xi) = \exp[-\Delta(q(z,x,\xi)+q(z,y,\xi))],$$

which satisfies Property (Q_L) by Corollary 2.20. We then obtain

$$\sigma\{p_\Delta(z,x,y,\xi)\} - \exp[-2\Delta q(z,x,\xi)] = \Delta\lambda_\Delta(z,x,\xi),$$

where $\lambda_\Delta(z,x,\xi)$ is bounded w.r.t. z and Δ with values in $S^0_\rho(X\times\mathbb{R}^n)$ by Lemma 2.22 (using $m=\rho-\delta$). By the Calderón-Vaillancourt Theorem (see [14, Chapter 7, Sections 1,2] or [25, Section XIII-2]), we shall obtain the desired estimate for $(\rho_\Delta(z,x,D_x)\circ\rho_\Delta(z,x,D_x)^*u,u)_{(L^2,L^2)}$ if we prove $\mathrm{Re}(\exp[-2\Delta q(z,x,D_x)]u,u)_{(L^2,L^2)} \leq (1+C\Delta)\|u\|^2_{L^2}$ for all $u\in\mathscr{S}(\mathbb{R}^n)$.

We set $r_\Delta(z,x,\xi) = (1 - \exp[-2\Delta q(z,x,\xi)])/\Delta$ for $\Delta > 0$ and observe that $r_\Delta(z,x,\xi)$ satisfies the conditions listed in Proposition 2.23 uniformly w.r.t. z and Δ. In fact, a first-order Taylor formula gives $\|r_\Delta(z,x,\xi)\| \le C\langle\xi\rangle$. By Property (Q_L) we obtain

$$\|\partial_x^\alpha r_\Delta(z,x,\xi)\| \le C\langle\xi\rangle, \quad |\alpha| = 1, \qquad \|\partial_\xi^\beta r_\Delta(z,x,\xi)\| \le C, \quad |\beta| = 1,$$

using $m = \rho$ in (Q_L) in both cases. Finally, if $|\alpha + \beta| = 2$, we obtain that $\partial_x^\alpha \partial_\xi^\beta r_\Delta(z,x,\xi)$ is bounded uniformly w.r.t. z and Δ with values in $S_\rho^{\rho-\delta+\delta|\alpha|-\rho|\beta|}(X \times \mathbb{R}^n)$ by choosing $m = \rho - \delta$ in (Q_L).

By Proposition 2.23 we thus obtain $\operatorname{Re}(r_\Delta(z,x,D_x)u,u)_{(L^2,L^2)} \ge -C\|u\|_{L^2}^2$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ which yields

$$\|u\|_{L^2}^2 - \operatorname{Re}(\exp[-2\Delta q(z,x,D_x)]u,u)_{(L^2,L^2)} \ge -C\Delta\|u\|_{L^2}^2,$$

which concludes the proof. ∎

We are now ready to give an estimate of the operator norm of the thin-slab propagator, $\mathcal{G}_{(z',z)}$, in $L(H^{(s)}(\mathbb{R}^n), H^{(s)}(\mathbb{R}^n))$ for any $s \in \mathbb{R}$.

Theorem 2.26. *Let $s \in \mathbb{R}$. There exists $M > 0$, $\Delta_5 > 0$ such that*

$$\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})} \le 1 + M\Delta,$$

for all $z', z \in [0,Z]$ such that $0 \le \Delta = z' - z \le \Delta_5$.

In the proof we assume that c_1 satisfies property (P_L) for some $L \ge 2$. We know that it is always true for $L = 2$ by Lemma 2.13 but special choices for c_1 can be made. As before we use $\rho = 1 - 1/L$ and $\delta = 1/L$ with $\rho > \delta$ for $L > 2$ and $\rho = \delta = \frac{1}{2}$ for $L = 2$. In the proof we proceed classically by computing $\mathcal{G}_{(z',z)} \circ \mathcal{G}^*_{(z',z)}$ and use the classical results on ψDOs (see e.g. [18, Section 5] and also [7]). Here we however do not content ourselves with the continuity of $\mathcal{G}_{(z',z)}$ but we want to obtain a precise estimate of the operator norm in $L(H^{(s)}(X), H^{(s)}(X'))$, which will be required in the sequel. Here we exploit the fact that Δ can be taken arbitrarily small which allows to carry out some explicit computations.

Proof. Let $s \in \mathbb{R}$, then the kernel of $\mathcal{A}_{(z',z)} := \mathcal{G}_{(z',z)} \circ E^{(-s)}$ is given by

$$\mathcal{A}_{(z',z)}(x',x) = \int \exp[i\phi_{(z',z)}(x',x,\xi)]\, g_{(z',z)}(x',\xi)\, \langle\xi\rangle^{-s}\, đ\xi.$$

Computing the kernel $D_{(z',z)}$ of $\mathcal{D}_{(z',z)} := \mathcal{A}_{(z',z)} \circ \mathcal{A}^*_{(z',z)}$ we obtain

$$D_{(z',z)}(x',x) = \int \exp\left[i\langle x' - x|\xi\rangle + i\Delta\left(b_1(z,x',\xi) - b_1(z,x,\xi)\right)\right] d_{(z',z)}(x',x,\xi)\, đ\xi$$

where

$$d_{(z',z)}(x',x,\xi) = \exp[-\Delta(c_1(z,x',\xi) + c_1(z,x,\xi)]\, g_{(z',z)}(x',\xi)\, \overline{g_{(z',z)}}(x,\xi)\, \langle\xi\rangle^{-2s}.$$

We write $b_1(z,x',\xi) - b_1(z,x,\xi) = \langle x' - x|h(z,x',x,\xi)\rangle$ where the function h is smooth, homogeneous of degree one in ξ, $|\xi| \ge 1$, and continuous w.r.t. z with values in $S^1(X' \times X \times \mathbb{R}^n)$ by Assumption 1.1 and estimate (1.1.9) in [10]. We thus obtain that the change of variables $\xi \to \xi + \Delta h(z, x', x, \xi) = H_{(\Delta,z,x',x)}(\xi)$ is a global diffeomorphism for Δ small enough (uniformly in $z \in [0, Z]$). Let $\tilde{\xi}(\Delta, z, x', x, \xi) = H^{-1}_{(\Delta,z,x',x)}(\xi)$. We thus have

$$D_{(z',z)}(x', x) = \int \exp\left[i\langle x' - x|\xi\rangle\right] d_{(z',z)}(x', x, \tilde{\xi}(\Delta, z, x', x, \xi))\, \mathcal{J}_\Delta(z, x', x, \xi)\, đ\xi$$

where $\mathcal{J}_\Delta(z, x', x, \xi)$ is the Jacobian.

Lemma 2.27. *The function $\tilde{\xi}(\Delta, z, x', x, \xi)$ is homogeneous of degree one in ξ, for $|\xi| \geq 1$, continuous w.r.t. z, $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^1(\mathbb{R}^{2n} \times \mathbb{R}^n)$ if Δ is small enough, i.e.,*

$$\exists \Delta_5 > 0, \quad \tilde{\xi} \in \mathscr{C}^0([0, Z], \mathscr{C}^\infty([0, \Delta_5], S^1(\mathbb{R}^{2n} \times \mathbb{R}^n))).$$

This lemma is in fact a variant of part of the results of Proposition 1.5 in [14, Chapter 10].

Proof. Homogeneity is clear. We have

$$\begin{aligned} |\tilde{\xi}(\Delta, z, x', x, \xi)| &= |\xi - \Delta h(z, x', x, \tilde{\xi}(\Delta, z, x', x, \xi))| \\ &\leq 1 + \Delta C(1 + |\tilde{\xi}(\Delta, z, x', x, \xi)|), \quad |\xi| = 1, \end{aligned}$$

which yields, because of homogeneity,

$$|\tilde{\xi}(\Delta, z, x', x, \xi)| \leq \frac{1 + \Delta C}{1 - \Delta C}(1 + |\xi|), \ |\xi| \geq 1,$$

for Δ small enough, uniformly chosen w.r.t. $z \in [0, Z]$, $x', x \in \mathbb{R}^n$. Differentiating the j^{th} coordinate of ξ,

$$\xi_j = \tilde{\xi}_j(\Delta, z, x', x, \xi) + \Delta h_j(z, x', x, \tilde{\xi}(\Delta, z, x', x, \xi)), \ j = 1, \ldots, n,$$

w.r.t. x_i yields

$$\begin{aligned} \partial_{x_i}\tilde{\xi}_j(\Delta, z, x', x, \xi) &+ \Delta \partial_{x_i} h_j(z, x', x, \tilde{\xi}(\Delta, z, x', x, \xi)) \\ &+ \Delta \sum_l \partial_{\tilde{\xi}_l} h_j(z, x', x, \tilde{\xi}(\Delta, z, x', x, \xi))\, \partial_{x_i}\tilde{\xi}_l(\Delta, z, x', x, \xi) = 0, \\ & \qquad j = 1, \ldots, n. \end{aligned} \tag{2.23}$$

The partial derivatives of h are bounded for $|\xi| = 1$. We can solve for $\partial_{x_i}\tilde{\xi}(\Delta, z, x', x, \xi)$ when Δ is sufficiently small and find the expected estimate from that obtained for $\tilde{\xi}(\Delta, z, x', x, \xi)$:

$$\exists C > 0, \ |\partial_{x_i}\tilde{\xi}(\Delta, z, x', x, \xi)| \leq C(1 + |\xi|), \ x', x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n.$$

Differentiating w.r.t. x'_i, ξ_i, and Δ yields similar structures and the proper symbol estimates. The proof carries on by induction. Note that the required size for Δ to solve the systems of the form (2.23) remains fixed along the induction process. ■

Continuation of the proof of Theorem 2.26. From (the proof of) Lemma 2.27 we also obtain that the Jacobian $\mathcal{J}_\Delta(z, x', x, \xi)$ is homogeneous of degree zero in ξ, $|\xi| \geq 1$, and is continuous w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^0(\mathbb{R}^{2n} \times \mathbb{R}^n)$.

We write $\tilde{p}_\Delta(z,x',x,\xi) := \exp[-\Delta(c_1(z,x',\xi)+c_1(z,x,\xi)]$. As c_1 satisfies Property (P_L) we then have $\tilde{p}_\Delta$ satisfying property (Q_L) by Corollary 2.20. Define $p_\Delta(z,x',x,\xi) := \tilde{p}_\Delta(z,x',x,\tilde{\xi}(\Delta,z,x',x,\xi))$. Since

$$\tilde{\xi}(\Delta,z,x',x,\xi) = \xi - \Delta h(z,x',x,\tilde{\xi}(\Delta,z,x',x,\xi))$$

Lemma 2.10 and Lemma 2.27 yield that $p_\Delta \in S^0_\rho(\mathbb{R}^{2n}\times\mathbb{R}^n)$ for Δ small enough. Lemma 2.18 yields that p_Δ satisfies Property (Q_L). We then have

$$d_{(z',z)}(x',x,\tilde{\xi}(\Delta,z,x',x,\xi))\,\mathcal{J}_\Delta(z,x',x,\xi) =: p_\Delta(z,x',x,\xi)\,k_\Delta(z,x',x,\xi),$$

where $k_\Delta(z,.)$ is bounded w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^{-2s}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ and $k_\Delta(z,x',x,\xi)|_{\Delta=0} = \langle\xi\rangle^{-2s}$ by Lemma 2.10 and Lemma 2.27. By Theorem 1.1.9 and formula (1.1.9) in [10] we obtain

$$k_\Delta(z,x',x,\xi) = \langle\xi\rangle^{-2s} + \Delta\tilde{k}_\Delta(z,x',x,\xi),$$

where $\tilde{k}_\Delta$ is bounded w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^{-2s}(\mathbb{R}^{2n}\times\mathbb{R}^n)$.

Call $\mathcal{F}_{(z',z)} := E^{(s)}\circ\mathcal{D}_{(z',z)}\circ E^{(s)}$. Its symbol is in $S^0_\rho(\mathbb{R}^n\times\mathbb{R}^n)$ and is given by

$$\begin{aligned} f_{(z',z)}(x',\xi) &:= (\langle\xi\rangle^s \,\#\, \sigma\{p_\Delta(z,x',x,\xi)\,k_\Delta(z,x',x,\xi)\} \,\#\, \langle\xi\rangle^s)(x',\xi)\\ &= (\langle\xi\rangle^s \,\#\, \sigma\left\{p_\Delta(z,x',x,\xi)\langle\xi\rangle^{-2s}\right\} \,\#\, \langle\xi\rangle^s)(x',\xi)\\ &\quad + \Delta(\langle\xi\rangle^s \,\#\, \sigma\left\{p_\Delta(z,x',x,\xi)\tilde{k}_\Delta(z,x',x,\xi)\right\} \,\#\, \langle\xi\rangle^s)(x',\xi). \end{aligned}$$

As p_Δ is bounded w.r.t. z and Δ, Δ small enough, with values in $S^0_\rho(\mathbb{R}^{2n}\times\mathbb{R}^n)$ (Property (Q_L)) we obtain that the second term in the equation above satisfies the same property and thus we can write

$$\mathcal{F}_{(z',z)} = \mathcal{F}^a_{(z',z)} + \Delta\mathcal{F}^1_{(z',z)},$$

where $\mathcal{F}^a_{(z',z)}$ has for symbol

$$(\langle\xi\rangle^s \,\#\, \sigma\left\{p_\Delta(z,x',x,\xi)\langle\xi\rangle^{-2s}\right\} \,\#\, \langle\xi\rangle^s)(x',\xi)$$

and $\|\mathcal{F}^1_{(z',z)}\|_{(L^2,L^2)} \le K^1$, uniformly in $z\in[0,Z]$ and Δ, Δ small enough, by the Calderón-Vaillancourt Theorem (see [14, Chapter 7, Sections 1,2] or [25, Section XIII-2]) in the case $L=2$ and by Theorem 18.1.11 in [8] in the case $L>2$. With Lemma 2.22 we see that

$$\sigma\left\{p_\Delta(z,x',x,\xi)\langle\xi\rangle^{-2s}\right\} - p_\Delta(z,x',x',\xi)\langle\xi\rangle^{-2s} = \Delta\lambda_\Delta(z,x',\xi),$$

where λ_Δ is bounded w.r.t. Δ and z with values in $S^{-2s}_\rho(\mathbb{R}^n\times\mathbb{R}^n)$. We thus obtain

$$\mathcal{F}^a_{(z',z)} = \mathcal{F}^b_{(z',z)} + \Delta\mathcal{F}^2_{(z',z)},$$

where $\mathcal{F}^b_{(z',z)}$ has for symbol

$$\begin{aligned} f^b_\Delta(z,x',\xi) &:= (\langle\xi\rangle^s \,\#\, p_\Delta(z,x',x',\xi)\langle\xi\rangle^{-2s} \,\#\, \langle\xi\rangle^s)(z,x',\xi)\\ &= (\langle\xi\rangle^s \,\#\, p_\Delta(z,x',x',\xi)\langle\xi\rangle^{-s})(z,x',\xi) \end{aligned}$$

and $\|\mathcal{F}^2_{(z',z)}\|_{(L^2,L^2)} \le K^2$ uniformly in $z\in[0,Z]$ and Δ, Δ small enough.

For the rest of the proof, if we don't write it explicitly, by p_Δ and $p_\Delta(z,x,\xi)$ we shall actually mean $p_\Delta(z,x,x,\xi)$.

Lemma 2.28.

$$(\langle . \rangle^s \# p_\Delta(z, .)\langle . \rangle^{-s})(z, x, \xi) - p_\Delta(z, x, \xi) = \Delta \mu_\Delta(z, x, \xi),$$

where $\mu_\Delta(z, x, \xi)$ is bounded w.r.t. z and Δ with values in $S^0_\rho(X \times \mathbb{R}^n)$.

Proof. We write

$$p_\Delta(z, x, \xi) = \langle \xi \rangle^{-s} \iint \exp[-i\langle y|\xi - \eta\rangle] \langle \eta \rangle^s p_\Delta(z, x, \xi) \, đ\eta \, dy$$

and thus obtain, with the oscillatory integral representation for the composition formula,

$$(\langle . \rangle^s \# p_\Delta(z, .)\langle . \rangle^{-s})(z, x, \xi) - p_\Delta(z, x, \xi) = \\ \langle \xi \rangle^{-s} \iint \exp[-i\langle y|\xi - \eta\rangle] \langle \eta \rangle^s (p_\Delta(z, x - y, \xi) - p_\Delta(z, x, \xi)) \, đ\eta \, dy.$$

With Taylor's formula and applying an integration by part, we find (we have supposed $n = 1$ for the sake of simplicity but it naturally extends to $n \geq 1$)

$$(\langle . \rangle^s \# p_\Delta(z, .)\langle . \rangle^{-s})(z, x, \xi) - p_\Delta(z, x, \xi) = \\ - \langle \xi \rangle^{-s} \int_0^1 \iint i \exp[-i\langle y|\xi - \eta\rangle] \, \partial_\eta \langle \eta \rangle^s \, \partial_x p_\Delta(z, x - ry, \xi) \, đ\eta \, dy \, dr.$$

Using Property (Q_L) with $m = 1 - \delta$ we find

$$(\langle . \rangle^s \# p_\Delta(z, .)\langle . \rangle^{-s})(z, x, \xi) - p_\Delta(z, x, \xi) = -\Delta \langle \xi \rangle^{-s} \\ \int_0^1 \iint i \exp[-i\langle y|\xi - \eta\rangle] \, \partial_\eta \langle \eta \rangle^s \, q^{m10}_\Delta(z, (1 - r)x + r(x - y), \xi) \, đ\eta \, dy \, dr \\ = -\Delta \langle \xi \rangle^{-s} (\partial_\xi \langle \xi \rangle^s \# \tilde{q}^{m10}_\Delta(z, u, x, \xi))|_{u=x},$$

where

$$\tilde{q}^{m10}_\Delta(z, u, x, \xi) = \int_0^1 q^{m10}_\Delta(z, (1 - r)u + rx, \xi) \, dr.$$

As $\tilde{q}^{m10}_\Delta$ is bounded w.r.t. Δ and z with values in $S^1_\rho(\mathbb{R}^{2n} \times \mathbb{R}^n)$ we obtain the result. ∎

End of the proof of Theorem 2.26. With the previous lemma we see that

$$\mathcal{F}^b_{(z',z)} = \mathcal{F}^c_{(z',z)} + \Delta \mathcal{F}^3_{(z',z)}$$

where $\mathcal{F}^c_{(z',z)}$ has for symbol $p_\Delta(z, x', x', \xi)$ and $||\mathcal{F}^3_{(z',z)}||_{(L^2,L^2)} \leq K^3$ uniformly in $z \in [0, Z]$ and Δ, Δ small enough.

From Theorem 2.25 we have $||\mathcal{F}^c_{(z',z)}||_{(L^2,L^2)} \leq 1 + \Delta K^4$, for some $K^4 \geq 0$. We thus obtain that $||\mathcal{F}_{(z',z)}||_{(L^2,L^2)} \leq 1 + \Delta K$ where $K = K^1 + K^2 + K^3 + K^4$. With the definition of $\mathcal{F}_{(z',z)}$ it follows that

$$||\mathcal{G}_{(z',z)}||_{(H^{(s)},H^{(s)})} = ||(\mathcal{G}_{(z',z)})^*||_{(H^{(s)},H^{(s)})} \leq \sqrt{1 + \Delta K},$$

which concludes the proof of Theorem 2.26. ∎

We observe that for Δ small enough, the function $\langle x'|\xi\rangle + \Delta b_1(z, x', \xi)$ satisfies the conditions (P)-(i), (P)-(ii), and (P)-(iii) in [13, page 2]. With Lemmas 2.13 and 2.17, we observe that an FIO with phase function $\phi_{(z',z)}(x', x, \xi)$ and amplitude in $\sigma_A(z, x', \xi)$ in $S^m(X\times\mathbb{R})$ may actually be understood as an FIO with real phase $\langle x'-x|\xi\rangle+\Delta b_1(z, x', \xi)$ and amplitude $\sigma_A(z, x', \xi)\exp[-\Delta c_1(z, x', \xi)]$ in $S^m_\rho(X\times\mathbb{R})$. Applying Theorem 2.5 and the following remark in [13] we obtain the following proposition.

Proposition 2.29. *Let $\mathcal{A}_{(z',z)}$ be the global FIO with kernel*

$$A_{(z',z)}(x', x) = \int \exp[i\phi_{(z',z)}(x', x, \xi)]\ \sigma_A(z, x', \xi)\ d\!\!{}^-\xi$$

with $\sigma_A(z, .)$ bounded w.r.t. z with values in $S^m(X\times\mathbb{R}^n)$, $m \in \mathbb{R}$. Then for all $s \in \mathbb{R}$ there exist $M = M(s, m) \geq 0$ and $\Delta_6 > 0$ such that

$$\|\mathcal{A}_{(z',z)}\|_{(H^{(s)},H^{(s-m)})} \leq M\ p(\sigma_A(z, .))$$

for all $z \in [0, Z]$, and $0 \leq \Delta \leq \Delta_6$, where $p(.)$ is some appropriately chosen seminorm in $S^m(X\times\mathbb{R}^n)$.

This proposition could also be proven by adapting the proof of Theorem 2.26 to this case. Note that in the case $\sigma_A = g_{(z',z)}$ we were able, in the proof of Theorem 2.26, to achieve a finer estimate. The proof heavily relies on the particular structure of the phase function and the amplitude that can be taken as “close” as we want to those of the identity operator by taking Δ small enough.

3 The approximation Ansatz. Convergence in Sobolev spaces

We first define the Ansatz that approximates the solution operator to (1.5)–(1.6). We choose to use a constant-step subdivision of the interval $[0, Z]$ but the method and results presented here can be naturally adapted to any subdivision of $[0, Z]$.

Definition 3.1. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\mathcal{W}_{\mathfrak{P},z}$ is defined as*

$$\mathcal{W}_{\mathfrak{P},z} := \begin{cases} \mathcal{G}_{(z,0)} & \text{if } 0 \leq z \leq z^{(1)}, \\ \mathcal{G}_{(z,z^{(k)})}\displaystyle\prod_{i=k}^{1}\mathcal{G}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \leq z \leq z^{(k+1)}. \end{cases}$$

The following proposition will be useful in the sequel.

Proposition 3.2. *Let $s \in \mathbb{R}$. There exists $K > 0$ such that for every subdivision $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ of $[0, Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ and $\mathcal{W}_{\mathfrak{P},z}$ as defined in Definition 3.1 we have*

$$\forall z \in [0, Z],\quad \|\mathcal{W}_{\mathfrak{P},z}\|_{(H^{(s)},H^{(s)})} \leq K,$$

if $\Delta_{\mathfrak{P}}$ is small enough.

Proof. By Theorem 2.26 there exits $M > 0$ such that if $\Delta = z' - z$ is small enough then $\|\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s)})} \leq 1 + \Delta M$ for all $z \in [0, Z]$; we then obtain $\|\mathcal{W}_{\mathfrak{P},z}\|_{(H^{(s)},H^{(s)})} \leq (1 + \Delta_{\mathfrak{P}} M)^N = (1 + \frac{ZM}{N})^N$ which is bounded as it converges to $\exp[ZM]$ as N goes to ∞. ∎

It should be first noticed that $\mathcal{W}_{\mathfrak{P},z}$ is not the solution to problem (1.5)–(1.6) even in the case where the symbols b and c only depend on the transversal variable, x. While singularities propagate along the bicharacteristics associated with $-\mathrm{Im}(a_1) = b_1$, observe however that, with the form of the phase function $\phi_{(z',z)}$ in (2.8), the operator $\mathcal{G}_{(z',z)}$ propagates singularities along straight lines. See Part II, for further details, in particular regarding the set $J_{(z',z)\mathbb{R}}$ that replaces the canonical relation for the propagation of singularities for FIOs with complex phase [9, Sections 25.4-5].

Furthermore, by composing the operators $\mathcal{G}_{(z'',z')}$ and $\mathcal{G}_{(z',z)}$, one convinces oneself that

$$\mathcal{G}_{(z'',z)} \neq \mathcal{G}_{(z'',z')} \circ \mathcal{G}_{(z',z)}$$

in general if $z'' \geq z' \geq z \in [0, Z]$ (use again that singularities propagate along straight lines). The family of operators $(\mathcal{G}_{(z',z)})_{(z',z)\in[0,Z]}$ is thus neither a semigroup nor an evolution system.

We now proceed towards the proof of the convergence of $\mathcal{W}_{\mathfrak{P},z}$ to the solution operator to problem (1.5)–(1.6) in the sense of Sobolev norms as $N = |\mathfrak{P}|$ goes to ∞.

Lemma 3.3. *Let $s \in \mathbb{R}$ and $z'', z \in [0, Z]$, with $z < z''$. The map $z' \mapsto \mathcal{G}_{(z',z)}$, for $z' \in [z, z'']$, is Lipschitz continuous with values in $L(H^{(s+1)}(X), H^{(s)}(X))$, for $z'' - z = \Delta$ small enough. More precisely there exists $C > 0$ such that for all $u_0 \in H^{(s+1)}(X)$ and $z^{(1)}, z^{(2)} \in [z, z'']$*

$$\|(\mathcal{G}_{(z^{(2)},z)} - \mathcal{G}_{(z^{(1)},z)})(u_0)\|_{H^{(s)}} \leq C|z^{(2)} - z^{(1)}|\|u_0\|_{H^{(s+1)}}. \tag{3.24}$$

Proof. Let $z^{(1)}, z^{(2)} \in [z'', z]$ and let $u_0 \in H^{s+1}(X)$. Write

$$(\mathcal{G}_{(z^{(2)},z)} - \mathcal{G}_{(z^{(1)},z)})(u_0)(x') = \\ -\int_{z^{(1)}}^{z^{(2)}} \iint \exp[i\langle x' - x|\xi\rangle - (z' - z)a(z, x', \xi)]\, a(z, x', \xi)\ \ u_0(x)\, dx\, đ\xi\, dz'.$$

When Δ is small enough we can apply Proposition 2.29 and obtain (3.24). ∎

Lemma 3.4. *Let $s \in \mathbb{R}$, $z'', z \in [0, Z]$, with $z < z''$, and let $u_0 \in H^{(s+1)}(X)$. Then the map $z' \mapsto \mathcal{G}_{(z',z)}(u_0)$ is in $\mathscr{C}^0([z, z''], H^{(s+1)}(X)) \cap \mathscr{C}^1([z, z''], H^{(s)}(X))$ for $z'' - z = \Delta$ small enough.*

Proof. Let $z^{(1)} \in [z, z'']$ and let $\varepsilon > 0$. Choose $z'' - z$ small enough such that Theorem 2.26 and Lemma 3.3 apply and choose $u_1 \in H^{(s+2)}$ such that $\|u_0 - u_1\|_{H^{(s+1)}} \leq \varepsilon$. Then for $z^{(2)} \in [z, z'']$

$$\begin{aligned} \|\mathcal{G}_{(z^{(2)},z)}(u_0) - \mathcal{G}_{(z^{(1)},z)}(u_0)\|_{H^{(s+1)}} &\leq \|\mathcal{G}_{(z^{(2)},z)}(u_0 - u_1)\|_{H^{(s+1)}} \\ &\quad + \|\mathcal{G}_{(z^{(2)},z)}(u_1) - \mathcal{G}_{(z^{(1)},z)}(u_1)\|_{H^{(s+1)}} + \|\mathcal{G}_{(z^{(1)},z)}(u_0 - u_1)\|_{H^{(s+1)}} \\ &\leq 2(1 + \Delta M)\varepsilon + C|z^{(2)} - z^{(1)}|\|u_1\|_{H^{(s+2)}}. \end{aligned} \tag{3.25}$$

The continuity of the map follows. Differentiating $\mathcal{G}_{(z',z)}(u_0)$ w.r.t. z' we can prove that the resulting map $z' \mapsto \partial_{z'}\mathcal{G}_{(z',z)}(u_0)$ is Lipschitz continuous with values in $L(H^{(s+2)}, H^{(s)})$ following the proof of Lemma 3.3: there exists $C > 0$ such that for all $v \in H^{(s+2)}(X)$

$$\|(\partial_{z'}\mathcal{G}_{(z^{(2)},z)} - \partial_{z'}\mathcal{G}_{(z^{(1)},z)})(v)\|_{H^{(s)}} \leq C|z^{(2)} - z^{(1)}|\|v\|_{H^{(s+2)}}.$$

We also see that the map $v \mapsto \partial_{z'} G_{(z',z)}(v)$ is continuous from $H^{(s+1)}$ into $H^{(s)}$ with bounded continuity module according to Proposition 2.29. With $u_0 \in H^{(s+1)}(X)$ we make a similar choice for $u_1 \in H^{(s+2)}(X)$ and obtain an estimate for

$$\|\partial_{z'} \mathcal{G}_{(z^{(2)},z)}(u_0) - \partial_{z'} \mathcal{G}_{(z^{(1)},z)}(u_0)\|_{H^{(s)}}$$

of the same form as in (3.25). ∎

The two previous lemmas yield the following proposition.

Proposition 3.5. *Let $s \in \mathbb{R}$, $\mathfrak{P}$ a subdivision of $[0, Z]$ as in Definition 3.1 and let $u_0 \in H^{(s+1)}(X)$. Then the map $\mathcal{W}_{\mathfrak{P},z}(u_0)$ is $\mathscr{C}^0([0, Z], H^{(s+1)}(X))$ and piecewise $\mathscr{C}^1([0, Z], H^{(s)}(X))$ if $\mathfrak{P}$ is chosen such that $\Delta_{\mathfrak{P}}$ is small enough. The map $z \mapsto \mathcal{W}_{\mathfrak{P},z}(u_0)$ is in fact globally Lipschitz with $C > 0$ such that*

$$\|\mathcal{W}_{\mathfrak{P},z'}(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{H^{(s)}} \leq C|z' - z| \|u_0\|_{H^{(s+1)}}.$$

We recall that $U(z', z)$ is the solution operator of the Cauchy problem (1.5)–(1.6). We can then apply the energy estimate (1.7) to $U(z, 0)(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)$ (adapt the proof of Lemma 23.1.1 in [8] to the case of a Lipschitz piecewise $\mathscr{C}^1$ function) and obtain

$$\begin{aligned} \text{(3.26)} \quad & \sup_{z\in[0,Z]} \exp[-\lambda z] \, \|U(z, 0)(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{H^{(s)}} \\ & \qquad \leq 2 \int_0^Z \exp[-\lambda z] \, \|(\partial_z + a_z(x, D_x))\mathcal{W}_{\mathfrak{P},z}(u_0)\|_{H^{(s)}} dz. \end{aligned}$$

Let $u_0 \in H^{(s+1)}(X)$ and let $\mathfrak{P} = \{z^{(0)}, \dots, z^{(N)}\}$. We take $z \in]z^{(k)}, z^{(k+1)}[$. Then

$$\begin{aligned} & (\partial_z + a_z(x, D_x)) \, \mathcal{W}_{\mathfrak{P},z}(u_0) \\ & \qquad = (\partial_z + a_z(x, D_x)) \left(\mathcal{G}_{(z,z^{(k)})} \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})}(u_0) \right) \\ & \qquad = (\partial_z + a_z(x, D_x)) \left(\mathcal{G}_{(z,z^{(k)})}(u_k) \right) \end{aligned}$$

with $u_k := \prod_{i=k}^{1} \mathcal{G}_{(z^{(i)},z^{(i-1)})}(u_0)$ which is in $H^{(s+1)}(X)$ by Theorem 2.26. We first turn our attention towards the term $(\partial_z + a_z(x, D_x)) \left(\mathcal{G}_{(z,z^{(k)})}(u) \right)$ for any $u \in H^{(s+1)}(X)$ as the norm of u_k in $H^{(s+1)}(X)$ remains under control even if $|\mathfrak{P}| = N$ becomes very large by Proposition 3.2:

$$\text{(3.27)} \qquad \exists K > 0, \;\; \|u_k\|_{H^{(s+1)}} \leq K \|u_0\|_{H^{(s+1)}}, \;\; k \in \{0, \dots, N\},$$

$N = |\mathfrak{P}| \in \mathbb{N}$, $u_0 \in H^{(s+1)}(X)$, if $\Delta_{\mathfrak{P}}$ is small enough.

We shall need the following lemma which is a variant to Lemma 2.22

Lemma 3.6. *Let $q_\Delta(z, x, y, \xi)$ be an amplitude in $S^0_\rho(\mathbb{R}^{2p} \times \mathbb{R}^p)$ depending on the parameters $\Delta \geq 0$ and $z \in [0, Z]$ that satisfies Property (Q_L) and such that $q_\Delta(z, .)|_{\Delta=0} = 0$. Let $r(x, y, \xi) \in S^s(\mathbb{R}^{2p} \times \mathbb{R}^p)$ for some $s \in \mathbb{R}$. Then*

$$\sigma\{q_\Delta \, r\}(z, x, \xi) - q_\Delta(z, x, x, \xi) \, r(x, x, \xi) = \Delta^{m+2\delta} \lambda^m_\Delta(z, x, \xi), \;\; 0 \leq m \leq \rho - \delta,$$

where $\lambda^m_\Delta(z, x, \xi)$ is bounded with respect to Δ and z with values in $S^{m+s-(\rho-\delta)}_\rho(\mathbb{R}^p \times \mathbb{R}^p)$.

Proof. We proceed as in the proof of Lemma 2.22 (we take $p = 1$ for the sake of concision). We obtain

$$\sigma\left\{q_\Delta r\right\}(z,x,\xi) - q_\Delta(z,x,x,\xi)r(x,x,\xi) = -\sigma\left\{ i\int_0^1 \partial_3\partial_4\lambda_\Delta(z,x,(1-s)x+sy,\xi)\,ds\right\},$$

where here

$$\begin{aligned}\partial_3\partial_4\lambda_\Delta(z,x,y,\xi) &= (\partial_y\partial_\xi q_\Delta)(z,x,y,\xi)\;r(x,y,\xi) + \partial_y q_\Delta(z,x,y,\xi)\partial_\xi r(x,y,\xi)\\ &\quad + \partial_\xi q_\Delta(z,x,y,\xi)\partial_y r(x,y,\xi) + q_\Delta(z,x,y,\xi)\partial_y\partial_\xi r(x,y,\xi).\end{aligned}$$

The first two terms are treated like in the proof of Lemma 2.22. For the third term, with Property (Q_L) we write

$$\partial_\xi q_\Delta\;\partial_y r = \Delta^{m'+\delta}\;q_\Delta^{m'(00)1}\;\partial_y r,\quad 0 \le m' \le 1-\delta,$$

where $q_\Delta^{m'(0,0)1}\;d_y r \in S_\rho^{m'+s-\rho}(\mathbb{R}^{2p}\times\mathbb{R}^p)$. We actually take $\delta \le m' \le 1-\delta$ and write $m = m'-\delta$. We obtain

$$\partial_\xi q_\Delta\;\partial_y r = \Delta^{m+2\delta}\;\tilde{q}_\Delta^m,$$

where $\tilde{q}_\Delta^m$ is bounded w.r.t. Δ with values in $S_\rho^{m+s-\rho+\delta}(\mathbb{R}^{2p}\times\mathbb{R}^p)$ and $0 \le m \le 1-2\delta = \rho-\delta$. For the fourth term we write

$$q_\Delta = \Delta^{m'}q_\Delta^{m'(0,0)0},\quad 0 \le m' \le 1,$$

where $q_\Delta^{m'(0,0)0} \in S_\rho^{m'}(\mathbb{R}^{2p}\times\mathbb{R}^p)$ by Property (Q_L) since $q_\Delta|_{\Delta=0} = 0$. We actually take $2\delta \le m' \le 1$ and write $m = m'-2\delta$. Then

$$q_\Delta\partial_y\partial_\xi r = \Delta^{m+2\delta}\hat{q}_\Delta^m,$$

where $\hat{q}_\Delta^m$ is bounded w.r.t. Δ with values in $S_\rho^{m+s-(\rho-\delta)}(\mathbb{R}^{2p}\times\mathbb{R}^p)$ as $m+s-1+2\delta = m+s-(\rho-\delta)$ and $0 \le m \le 1-2\delta = \rho-\delta$. We conclude like in the proof of Lemma 2.22. ■

For the next proposition we shall need the following assumption as announced in Section 1

Assumption 3.7. *The symbol $a(z,.)$ is assumed to be in $\mathscr{L}([0,Z],S^1(\mathbb{R}^n\times\mathbb{R}^n))$, i.e., Lipschitz continuous w.r.t. z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$, in the sense that,*

$$a(z',x,\xi) - a(z,x,\xi) = (z'-z)\tilde{a}(z',z,x,\xi),\quad 0 \le z \le z' \le Z$$

with $\tilde{a}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$.

Proposition 3.8. *Let $s \in \mathbb{R}$. There exists $\Delta_7 > 0$ and $C \ge 0$ such that for $z'-z = \Delta$, $\Delta \in [0,\Delta_7]$,*

$$\|(\partial_{z'} + a_{z'}(x,D_x))\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \le C\Delta^{\frac{1}{2}}.$$

Like in the proof of Theorem 2.26 we assume that c_1 satisfies property (P_L) for some $L \ge 2$. We know that it is always true for $L = 2$ by Lemma 2.13 but special choices for c_1 can be made. As before we use $\rho = 1-1/L$ and $\delta = 1/L$ with $\rho > \delta$ for $L > 2$ and $\rho = \delta = \frac{1}{2}$ for $L = 2$.

Proof. With Assumption 3.7 and Theorem 2.26, we have

$$\|(a_z(x, D_x) - a_{z'}(x, D_x))\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \leq C\Delta.$$

It is thus sufficient to prove

$$\|(\partial_{z'} + a_z(x, D_x))\mathcal{G}_{(z',z)}\|_{(H^{(s)},H^{(s-1)})} \leq C\Delta^{\frac{1}{2}}.$$

Let $\mathcal{A}_{(z',z)}$ be $\partial_{z'}\mathcal{G}_{(z',z)}$ and $\mathcal{B}_{(z',z)}$ be $a_z(x, D_x) \circ \mathcal{G}_{(z',z)}$ with respective kernels $A_{(z',z)}(x', x)$ and $B_{(z',z)}(x', x)$. We have

$$A_{(z',z)}(x', x) = -\int \exp[i\langle x' - x|\xi\rangle] \exp[-\Delta a(z, x', \xi)]\, a(z, x', \xi)\, đ\xi.$$

Let us define

$$\mathcal{D}_{(z',z)} := (\mathcal{A}_{(z',z)} + \mathcal{B}_{(z',z)}) \circ E^{-2s} \circ (\mathcal{A}_{(z',z)} + \mathcal{B}_{(z',z)})^*.$$

We prove in the following lemma that for $r, s \in \mathbb{R}$, $\|\mathcal{D}_{(z',z)}\|_{(H^{(r)},H^{(r+2s-2)})} \leq C\Delta$ uniformly w.r.t. $z \in [0, Z]$ for Δ small enough. The conclusion then follows: if $C_{(z',z)} := E^{s-1} \circ \mathcal{D}_{(z',z)} \circ E^{s-1}$ then $\|C_{(z',z)}\|_{(L^2,L^2)} \leq C\Delta$ (take $r = -s + 1$); then $\|E^{s-1} \circ (\mathcal{A}_{(z',z)} + \mathcal{B}_{(z',z)}) \circ E^{-s}\|_{(L^2,L^2)} \leq C\Delta^{\frac{1}{2}}$. ∎

Lemma 3.9. *Let $r, s \in \mathbb{R}$. Then $\|\mathcal{D}_{(z',z)}\|_{(H^{(r)},H^{(r+2s-2)})} \leq C\Delta$ uniformly w.r.t. $z \in [0, Z]$ for Δ small enough.*

Proof. The operator $\mathcal{D}_{(z',z)}$ is made up of four terms:

$$\mathcal{D}_{1,(z',z)} := \mathcal{A}_{(z',z)} \circ E^{-2s} \circ \mathcal{A}^*_{(z',z)}, \quad \mathcal{D}_{2,(z',z)} := \mathcal{A}_{(z',z)} \circ E^{-2s} \circ \mathcal{B}^*_{(z',z)},$$
$$\mathcal{D}_{3,(z',z)} := \mathcal{B}_{(z',z)} \circ E^{-2s} \circ \mathcal{A}^*_{(z',z)}, \quad \mathcal{D}_{4,(z',z)} := \mathcal{B}_{(z',z)} \circ E^{-2s} \circ \mathcal{B}^*_{(z',z)}.$$

The kernel of $\mathcal{D}_{1,(z',z)}$ is given by

$$D_{1,(z',z)}(x', x) = \int \exp\left[i\langle x' - x|\xi\rangle + i\Delta\left(b_1(z, x', \xi) - b_1(z, x, \xi)\right)\right]\, \tilde{d}_{1,z}(x', x, \xi)\, đ\xi,$$

where

$$\tilde{d}_{1,z}(x', x, \xi) = \omega_{(z',z)}(x', x, \xi)\, a(z, x', \xi)\, \overline{a}(z, x, \xi)$$

and

$$\omega_{(z',z)}(x', x, \xi) := g_{(z',z)}(x', \xi)\, \overline{g_{(z',z)}}(x, \xi) \exp[-\Delta(c_1(z, x', \xi) + c_1(z, x, \xi))]\langle\xi\rangle^{-2s},$$

with $g_{(z',z)}$ given in (2.9). Following the proof of Theorem 2.26 we write $b_1(z, x', \xi) - b_1(z, x, \xi) = \langle x' - x|h(z, x', x, \xi)\rangle$ where h is homogeneous of degree one in ξ, $|\xi| \geq 1$. The function h is continuous w.r.t. z with values in $S^1(X' \times X \times \mathbb{R}^n)$. We thus obtain that the change of variables $\xi \to \xi + \Delta h(z, x', x, \xi)$ is a global diffeomorphism for Δ small enough (uniformly in $z \in [0, Z]$). The Jacobian $\mathcal{J}_\Delta(z, x', x, \xi)$ is homogeneous of degree zero in ξ, $\mathscr{C}^\infty$ w.r.t. Δ and bounded w.r.t. z with values in $S^0(\mathbb{R}^{2n} \times \mathbb{R}^n)$. We then have

$$D_{1,(z',z)}(x', x) = \int \exp\left[i\langle x' - x|\xi\rangle\right] \tilde{d}_{1,z}(x', x, \tilde{\xi}(\Delta, \xi))\, \mathcal{J}(\Delta, z, x', x, \xi)\, đ\xi.$$

The function $\tilde{\xi}(\Delta, z, x', x, \xi)$, written $\tilde{\xi}(\Delta, \xi)$ for concision, is bounded w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ in $S^1(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and homogeneous of degree one in ξ as shown in Lemma 2.27. It follows that $\tilde{d}_{1,z}(x', x, \tilde{\xi}(\Delta, \xi))\ \mathcal{J}(\Delta, z, x', x, \xi)$ is bounded w.r.t. z and Δ with values in $S^{2-2s}_\rho(\mathbb{R}^{2n} \times \mathbb{R}^n)$ by Lemma 2.10 and the proof of Theorem 2.26. Note that if $\Delta = 0$ then $\tilde{\xi}(\Delta, \xi) = \xi$. The operator $\mathcal{D}_{1,(z',z)}$ is thus in Ψ^{2-2s}_ρ with symbol

$$d_{1,(z',z)}(x', \xi) = \sigma \left\{ \tilde{d}_{1,z}(x', x, \tilde{\xi}(\Delta, \xi))\ \mathcal{J}(\Delta, z, x', x, \xi) \right\} (x', \xi).$$

Similarly we prove that $\mathcal{A}_{(z',z)} \circ E^{-2s} \circ \mathcal{G}^*_{(z',z)}$ is the ψDO with *amplitude*

$$-\omega_{(z',z)}(x', x, \tilde{\xi}(\Delta, \xi))\ a(z, x', \tilde{\xi}(\Delta, \xi))\ \mathcal{J}(\Delta, z, x', x, \xi).$$

The operator $\mathcal{D}_{2,(z',z)}$ is thus in $\Psi^{2-2s}_\rho(X)$ with symbol

$$d_{2,(z',z)}(x', \xi) = -\sigma \left\{ \omega_{(z',z)}(x', x, \tilde{\xi}(\Delta, \xi))\ a(z, x', \tilde{\xi}(\Delta, \xi)) \mathcal{J}(\Delta, z, x', x, \xi) \right\}\ \#\ a^*(z, x', \xi).$$

Similarly we find that the operators $\mathcal{D}_{3,(z',z)}$ and $\mathcal{D}_{4,(z',z)}$ are in $\Psi^{2-2s}_\rho(X)$ with respective symbols

$$d_{3,(z',z)}(x', \xi) = -a(z, x', \xi)\ \#\ \sigma \left\{ \omega_{(z',z)}(x', x, \tilde{\xi}(\Delta, \xi))\ \mathcal{J}(\Delta, z, x', x, \xi)\ \overline{a}(z, x, \tilde{\xi}(\Delta, \xi)) \right\}$$

and

$$d_{4,(z',z)}(x', \xi) = a(z, x', \xi)\ \#\ \sigma \left\{ \omega_{(z',z)}(x', x, \tilde{\xi}(\Delta, \xi))\ \mathcal{J}(\Delta, z, x', x, \xi) \right\}\ \#\ a^*(z, x', \xi).$$

For $q(x', x, \xi)$ an amplitude we define

$$\begin{aligned} \Sigma\{q\}(x', \xi) := \sigma\{\langle\xi\rangle^{-2s}\ a(z, x', \xi)\ q(x', x, \xi)\ \overline{a}(z, x, \xi)\} \\ - \sigma\{\langle\xi\rangle^{-2s}\ a(z, x', \xi)\ q(x', x, \xi)\}\ \#\ a^*(z, x', \xi) \\ + a(z, x', \xi)\ \#\ \sigma\{\langle\xi\rangle^{-2s}\ q(x', x, \xi)\} \#\ a^*(z, x', \xi) \\ - a(z, x', \xi)\ \#\ \sigma\{\langle\xi\rangle^{-2s}\ q(x', x, \xi)\ \overline{a}(z, x, \xi)\}. \end{aligned}$$

The operator $\mathcal{D}_{(z',z)}$ is thus in $\Psi^{2-2s}_\rho(X)$ with symbol

$$d_{(z',z)} = d_{1,(z',z)} + d_{2,(z',z)} + d_{3,(z',z)} + d_{4,(z',z)}.$$

Such a symbol is bounded w.r.t. Δ, for Δ small enough, as the composition formula for symbols is a bounded map. Note that

$$g_{(z',z)}(x', \tilde{\xi}(\Delta, \xi))\ \overline{g_{(z',z)}}(x, \tilde{\xi}(\Delta, \xi))\ \langle\tilde{\xi}(\Delta, \xi)\rangle^{-2s}\ \mathcal{J}(\Delta, z, x', x, \xi) = \langle\xi\rangle^{-2s} + \Delta k_\Delta(z, x', x, \xi),$$

with the function k_Δ bounded w.r.t. z and $\mathscr{C}^\infty$ w.r.t. Δ with values in $S^{-2s}(X' \times X \times \mathbb{R}^n)$ as $g_{(z',z)}(x', \tilde{\xi}(\Delta, \xi))\ \overline{g_{(z',z)}}(x, \tilde{\xi}(\Delta, \xi))\langle\tilde{\xi}(\Delta, \xi)\rangle^{-2s}\ \mathcal{J}(\Delta, z, x', x, \xi)$ is itself $\mathscr{C}^\infty$ w.r.t. Δ by Lemma 2.10 (case $\rho = 1$) and equal to $\langle\xi\rangle^{-2s}$ when $\Delta = 0$. With a similar reasoning on $a_z(x', \tilde{\xi}(\Delta, \xi))$ and $a_z(x, \tilde{\xi}(\Delta, \xi))$ we thus obtain

$$\mathcal{D}_{(z',z)} = \mathcal{D}^a_{(z',z)} + \Delta\mathcal{D}^1_{(z',z)},$$

with symbols

$$d^a_{(z',z)} := \Sigma\{p_\Delta(z, x', x, \xi)\}$$

and $d^1_{(z',z)}$ which is bounded w.r.t. z and Δ with values in $S^{2-2s}_\rho(X'\times X\times\mathbb{R}^n)$. The symbol p_Δ was defined in the proof of Theorem 2.26 as

$$
\begin{aligned}
p_\Delta(z,x',x,\xi) &= \tilde{p}_\Delta(z,x',x,\tilde{\xi}(\Delta,z,x',x,\xi)) \\
&= \exp[-\Delta(c_1(z,x',\tilde{\xi}(\Delta,z,x',x,\xi)) + c_1(z,x,\tilde{\xi}(\Delta,z,x',x,\xi)))].
\end{aligned}
$$

Recall that it satisfies Property (Q_L) by Lemma 2.18.

The Calderón-Vaillancourt theorem (see [14, Chapter 7, Sections 1,2] or [25, Section XIII-2]) in the case $L = 2$ or Theorem 18.1.11 in [8] in the case $L > 2$ yield $\|\mathcal{D}^1_{(z',z)}\|_{(H^{(r)},H^{(r+2s-2)})} \le K_1$. Note that for a symbol $q(x',\xi)$ we have $\Sigma\{q(x',\xi)\} = 0$ since

$$
\sigma\{q(x',\xi)\,\overline{a}(z,x,\xi)\} = q(x',\xi)\ \#\ a^*(z,x',\xi) = \sigma\{q(x',\xi)\}\ \#\ a^*(z,x',\xi),
$$

for any symbol q. Thus $d^a(z',z) = \Sigma\{p_\Delta(z,x',x,\xi) - 1\}$. Lemma 3.6 allows us to write (take $m = \rho - \delta$)

$$
\begin{aligned}
&\sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x,\xi) - 1)a(z,x',\xi)\overline{a}(z,x,\xi)\} \\
&\qquad = \langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)a(z,x',\xi)\overline{a}(z,x',\xi) + \Delta\lambda_{\Delta,1}(z,x',\xi),
\end{aligned}
$$

where $\lambda_{\Delta,1}$ is bounded w.r.t. z and Δ with values in $S^{2-2s}_\rho(X'\times\mathbb{R}^n)$. We also write

$$
\begin{aligned}
&\sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x,\xi) - 1)a(z,x',\xi)\}\ \#\ a^*(z,x',\xi) \\
&\qquad = (\langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)a(z,x',\xi))\ \#\ a^*(z,x',\xi) \\
&\qquad\qquad + \Delta\lambda_{\Delta,2}(z,x',\xi)\ \#\ a^*(z,x',\xi) \\
&\qquad = \sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)a(z,x',\xi)\overline{a}(z,x,\xi)\} \\
&\qquad\qquad + \Delta\lambda_{\Delta,2}(z,x',\xi)\ \#\ a^*(z,x',\xi) \\
&\qquad = \langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)a(z,x',\xi)\overline{a}(z,x',\xi) \\
&\qquad\qquad + \Delta(\lambda_{\Delta,3}(z,x',\xi) + \lambda_{\Delta,2}(z,x',\xi)\ \#\ a^*(z,x',\xi)),
\end{aligned}
$$

where $\lambda_{\Delta,2}$ and $\lambda_{\Delta,3}$ are bounded w.r.t. z and Δ with values in $S^{1-2s}_\rho(X'\times\mathbb{R}^n)$ and $S^{2-2s}_\rho(X'\times\mathbb{R}^n)$, respectively. Similarly we have

$$
\begin{aligned}
&\sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x,\xi) - 1)\}\ \#\ a^*(z,x',\xi) \\
&\quad = (\langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1))\ \#\ a^*(z,x',\xi) + \Delta\lambda_{\Delta,4}(z,x',\xi)\ \#\ a^*(z,x',\xi) \\
&\quad = \sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)\overline{a}(z,x,\xi)\} + \Delta\lambda_{\Delta,4}(z,x',\xi)\ \#\ a^*(z,x',\xi) \\
&\quad = \langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)\overline{a}(z,x',\xi) \\
&\qquad + \Delta(\lambda_{\Delta,5}(z,x',\xi) + \lambda_{\Delta,4}(z,x',\xi)\ \#\ a^*(z,x',\xi)),
\end{aligned}
$$

where $\lambda_{\Delta,4}$ and $\lambda_{\Delta,5}$ are bounded w.r.t. z and Δ with values in $S^{-2s}_\rho(X'\times\mathbb{R}^n)$ and $S^{1-2s}_\rho(X'\times\mathbb{R}^n)$, respectively, and

$$
\begin{aligned}
&\sigma\{\langle\xi\rangle^{-2s}(p_\Delta(z,x',x,\xi) - 1)\overline{a}(z,x,\xi)\} \\
&\qquad = \langle\xi\rangle^{-2s}(p_\Delta(z,x',x',\xi) - 1)\overline{a}(z,x',\xi) + \Delta\lambda_{\Delta,6}(z,x',\xi),
\end{aligned}
$$

where $\lambda_{\Delta,6}$ is bounded w.r.t. z and Δ with values in $S^{1-2s}_\rho(X'\times\mathbb{R}^n)$. We thus obtain

$$
d^a_{(z',z)} = \Delta(\lambda_{\Delta,1} + \lambda_{\Delta,3} + \lambda_{\Delta,2}\ \#\ a^* + a\ \#\ \lambda_{\Delta,5} + a\ \#\ \lambda_{\Delta,4}\ \#\ a^*) = \Delta\tilde{d}^a_{(z',z)}
$$

with $\tilde{d}^a_{(z',z)}$ bounded w.r.t. z and Δ with values in $S^{2-2s}_\rho(X'\times\mathbb{R}^n)$. This concludes the proof. ■

We have thus obtained a convergence result in the Sobolev space $H^{(s)}(\mathbb{R}^n)$ for $\mathcal{W}_{\mathfrak{P},z}(u_0)$ if the initial data u_0 is in $H^{(s+1)}(\mathbb{R}^n)$. The result is actually the convergence of the Ansatz $\mathcal{W}_{\mathfrak{P},z}$ to the solution operator $U(z,0)$ in the norm of $L(H^{(s+1)}(\mathbb{R}^n), H^{(s)}(\mathbb{R}^n))$.

Theorem 3.10. *Assume that $a(z,.)$ is in $\mathscr{L}([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, i.e., Lipschitz continuous w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$, in the sense that,*

$$a(z', x, \xi) - a(z, x, \xi) = (z' - z)\tilde{a}(z', z, x, \xi), \quad 0 \le z \le z' \le Z$$

with $\tilde{a}(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$. Let $s \in \mathbb{R}$. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ converges to the solution operator $U(z,0)$ of the Cauchy problem (1.5)–(1.6) in $L(H^{(s+1)}(\mathbb{R}^n), H^{(s)}(\mathbb{R}^n))$ uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to zero with a convergence rate of order $\frac{1}{2}$:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{(H^{(s+1)},H^{(s)})} \le C\Delta_{\mathfrak{P}}^{\frac{1}{2}}, \quad z \in [0,Z].$$

Proof. Using (3.26) and (3.27) we obtain

$$\sup_{z\in[0,Z]} \exp[-\lambda z] \, \|U(z,0)(u_0) - \mathcal{W}_{\mathfrak{P},z}(u_0)\|_{H^{(s)}}$$
$$\le 2\int_0^Z \exp[-\lambda z] \, \Delta_{\mathfrak{P}}^{\frac{1}{2}} CK \|u_0\|_{H^{(s+1)}} dz \le C\Delta_{\mathfrak{P}}^{\frac{1}{2}} \|u_0\|_{H^{(s+1)}}.$$

The result follows. ∎

If we change the assumption made on the symbol $a(z,.)$ to some Hölder type continuity, then the corresponding change in the proof of Lemma 3.8 yields the following result.

Theorem 3.11. *Assume that $a(z,.)$ is in $\mathscr{C}^{0,\alpha}([0,Z], S^1(\mathbb{R}^n \times \mathbb{R}^n))$, i.e., Hölder continuous w.r.t. z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$, in the sense that, for some $0 < \alpha < 1$*

$$a(z', x, \xi) - a(z, x, \xi) = (z' - z)^{\alpha} \, \tilde{a}(z', z, x, \xi), \quad 0 \le z \le z' \le Z$$

with $\tilde{a}(z', z, x, \xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n \times \mathbb{R}^n)$. Let $s \in \mathbb{R}$. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ converges to the solution operator $U(z,0)$ of the Cauchy problem (1.5)–(1.6) in $L(H^{(s+1)}(\mathbb{R}^n), H^{(s)}(\mathbb{R}^n))$ uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to 0 with a convergence rate of order β:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{(H^{(s+1)},H^{(s)})} \le C\Delta_{\mathfrak{P}}^{\beta}, \quad z \in [0,Z],$$

with $\beta = \alpha$ for $0 < \alpha \le \frac{1}{2}$ and $\beta = \frac{1}{2}$ for $\frac{1}{2} \le \alpha < 1$.

A result similar to that of the previous theorems can be obtained with weaker assumptions, namely without assumptions on the symbol $a(z,.)$ like those made in Theorems 3.10 and 3.11, by introducing another, yet natural, Ansatz to approximate the exact solution to the Cauchy problem (1.5)–(1.6). For a symbol $q(z,y,\eta) \in \mathscr{C}^0([0,Z], S^m(\mathbb{R}^p \times \mathbb{R}^r))$ we define $\hat{q}_{(z',z)}(y,\eta) \in \mathscr{C}^0([0,Z]^2, S^m(\mathbb{R}^p \times \mathbb{R}^r))$

$$\hat{q}_{(z',z)}(y,\eta) := \frac{1}{z'-z}\int_z^{z'} q(s,y,\eta)\, ds.$$

Then we define

$$
\begin{aligned}
(3.28)\quad \hat{\phi}_{(z',z)}(x',x,\xi) &:= \langle x'-x|\xi\rangle + i\Delta\hat{a}_{1(z',z)}(x',\xi) \\
&= \langle x'-x|\xi\rangle + \Delta\hat{b}_{1(z',z)}(x',\xi) + i\Delta\hat{c}_{1(z',z)}(x',\xi)
\end{aligned}
$$

and

$$
(3.29)\qquad \hat{g}_{(z',z)}(x,\xi) := \exp[-\Delta\hat{a}_{0(z',z)}(x,\xi)]
$$

and finally, following [17], we denote by $\widehat{\mathcal{G}}_{(z',z)}$ the FIO with distribution kernel

$$
\begin{aligned}
\widehat{G}_{(z',z)}(x',x) &= \int \exp[i\langle x'-x|\xi\rangle]\exp[-\Delta\hat{a}_{(z',z)}(x',\xi)]\, đ\xi \\
&= \int \exp[i\hat{\phi}_{(z',z)}(x',x,\xi)]\,\hat{g}_{(z',z)}(x',\xi)\, đ\xi,
\end{aligned}
$$

with the associated approximation Ansatz in the following definition.

Definition 3.12. *Let $\mathfrak{P} = \{z^{(0)}, z^{(1)}, \ldots, z^{(N)}\}$ be a subdivision of $[0,Z]$ with $0 = z^{(0)} < z^{(1)} < \cdots < z^{(N)} = Z$ such that $z^{(i+1)} - z^{(i)} = \Delta_{\mathfrak{P}}$. The operator $\widehat{\mathcal{W}}_{\mathfrak{P},z}$ is defined as*

$$
\widehat{\mathcal{W}}_{\mathfrak{P},z} := \begin{cases} \widehat{\mathcal{G}}_{(z,0)} & \text{if } 0 \le z \le z^{(1)}, \\ \widehat{\mathcal{G}}_{(z,z^{(k)})} \prod_{i=k}^{1} \widehat{\mathcal{G}}_{(z^{(i)},z^{(i-1)})} & \text{if } z^{(k)} \le z \le z^{(k+1)}. \end{cases}
$$

Most results of Sections 2 and 3 apply to this new Ansatz. We give some details about how to adapt some of the proofs. We have the following lemma.

Lemma 3.13. *Let $q(z,y,\eta) \in \mathscr{C}^0([0,Z], S^1(\mathbb{R}^p \times \mathbb{R}^r))$ that satisfies Property (P_L). Then $\hat{q}_{(z',z)}(y,\eta)$ also satisfies Property (P_L).*

Property (P_L) in Definition 2.11 is now to be understood w.r.t. to two parameters z' and z.

Proof. Uniform bounds w.r.t. z and z' will be immediate. The case $|\alpha| + |\beta| \ge L$ is clear by Remark 2.12. Let then $|\alpha| + |\beta| < L$ and observe that

$$
\begin{aligned}
|\partial_y^\alpha \partial_\eta^\beta \hat{q}_{(z',z)}(y,\eta)| &= \Big|\frac{1}{z'-z}\int_z^{z'} \partial_y^\alpha d_\eta^\beta q(s,y,\eta)\, ds\Big| \\
&\le C(1+|\eta|)^{-|\beta|+(|\alpha|+|\beta|)/L}\,\frac{1}{z'-z}\int_z^{z'}(1+q(z,y,\eta))^{1-(|\alpha|+|\beta|)/L} ds \\
&\le C(1+|\eta|)^{-|\beta|+(|\alpha|+|\beta|)/L}\left(1+\frac{1}{z'-z}\int_z^{z'} q(z,y,\eta) ds\right)^{1-(|\alpha|+|\beta|)/L} \\
&= C(1+|\eta|)^{-|\beta|+(|\alpha|+|\beta|)/L}(1+\hat{q}_{(z',z)}(y,\eta))^{1-(|\alpha|+|\beta|)/L},
\end{aligned}
$$

by Jensen inequality as $t \mapsto -(1+t)^{1-(|\alpha|+|\beta|)/L}$ is convex when $|\alpha| + |\beta| < L$. ∎

As a consequence of Lemma 2.17 we have the following lemma.

Lemma 3.14. *Let $q(z, y, \eta) \in \mathscr{C}^0([0, Z], S^1(\mathbb{R}^p \times \mathbb{R}^r))$ that satisfies Property (P_L). Then $\hat{\rho}_\Delta := \exp[-\Delta \hat{q}_{(z',z)}(y, \eta)]$ satisfies Property (Q_L).*

The result of Theorem 2.26 thus applies to the modified thin-slab propagator $\widehat{\mathcal{G}}_{(z',z)}$ (Lemma 2.27 has to be slightly modified). The proof of Lemma 3.3 applies with the aid of Proposition 2.29 as

$$(\widehat{\mathcal{G}}_{(z^{(2)},z)} - \widehat{\mathcal{G}}_{(z^{(1)},z)})(u_0)(x') = \\ -\int_{z^{(1)}}^{z^{(2)}} \iint \exp[i\langle x' - x|\xi\rangle - (z' - z)\hat{a}_{(z',z)}(x', \xi)]\, a(z', x', \xi)\;\; u_0(x)\, dx\, đ\xi\, dz'.$$

To adapt the proof of Lemma 3.4 we need

Lemma 3.15. *Let $s \in \mathbb{R}$ and $z'', z \in [0, Z]$. The map $z' \mapsto \partial_{z'} \widehat{\mathcal{G}}_{(z',z)}$, for $z' \in [z'', z]$, is continuous with values in $L(H^{(s+2)}(X), H^{(s)}(X))$, for $z'' - z = \Delta$ small enough.*

Proof. We choose $\Delta = z'' - z$ sufficiently small such that the results of Section 1 apply. Let $z^{(1)}, z^{(2)} \in [z, z'']$. Then we have

$$\begin{aligned} \partial_{z'} \widehat{G}_{(z^{(2)},z)}(x', x) - \partial_{z'} \widehat{G}_{(z^{(1)},z)}(x', x) \\ = -\int \exp[i\langle x' - x|\xi\rangle] \Big(a(z^{(2)}, x', \xi) \exp[-{\textstyle\int_z^{z^{(2)}}} a(s, x', \xi)\, ds] \\ - a(z^{(1)}, x', \xi) \exp[-{\textstyle\int_z^{z^{(1)}}} a(s, x', \xi)\, ds] \Big) đ\xi \\ = A_{(z^{(2)},z^{(1)},z)}(x', x) + B_{(z^{(2)},z^{(1)},z)}(x', x), \end{aligned}$$

where

$$A_{(z^{(2)},z^{(1)},z)}(x', x) := -\int \exp[i\langle x' - x|\xi\rangle]\, a(z^{(2)}, x', \xi) \\ \Big(\exp[-{\textstyle\int_z^{z^{(2)}}} a(s, x', \xi)\, ds] - \exp[-{\textstyle\int_z^{z^{(1)}}} a(s, x', \xi)\, ds] \Big)\; đ\xi,$$

and

$$B_{(z^{(2)},z^{(1)},z)}(x', x) := -\int \exp[i\langle x' - x|\xi\rangle] \\ (a(z^{(2)}, x', \xi) - a(z^{(1)}, x', \xi))\;\; \exp[-{\textstyle\int_z^{z^{(1)}}} a(s, x', \xi)\, ds]\, đ\xi.$$

We write

$$A_{(z^{(2)},z^{(1)},z)}(x', x) = \int_{z^{(1)}}^{z^{(2)}} \int \exp[i\langle x' - x|\xi\rangle] \\ a(z^{(2)}, x', \xi)\, a(s, x', \xi)\;\; \exp[-(s - z)\hat{a}_{(s,z)}(x', \xi)]\, ds\, đ\xi$$

and for the associated operator, $\mathcal{A}_{(z^{(2)},z^{(1)},z)}$, we obtain by Proposition 2.29 that

$$\|\mathcal{A}_{(z^{(2)},z^{(1)},z)}\|_{(H^{(s+2)},H^{(s)})} \leq C|z^{(2)} - z^{(1)}|.$$

For the second term we can apply Proposition 2.29 which gives the estimate, for the associated operator, $\|\mathcal{B}_{(z^{(2)},z^{(1)},z)}(x',x)\|_{(H^{(s+2)},H^{(s)})} \leq C\ p(a(z^{(2)},.) - a(z^{(1)},.))$, with p a seminorm in $S^1(X\times\mathbb{R}^n)$. The continuity of $z\mapsto a(z,.)$ in $S^1(X\times\mathbb{R}^n)$ (Assumption 1.1) yields the result. ■

With the previous lemma we can easily adapt the proof of Lemma 3.4 and obtain the same result for $\widehat{\mathcal{G}}_{(z',z)}$.

Lemma 3.16. *Let $s\in\mathbb{R}$, $z'',z\in[0,Z]$, with $z<z''$, and let $u_0\in H^{(s+1)}(X)$. Then the map $z'\mapsto\widehat{\mathcal{G}}_{(z',z)}(u_0)$ is in $\mathscr{C}^0([z,z''],H^{(s+1)}(X))\cap\mathscr{C}^1([z,z''],H^{(s)}(X))$ for $z''-z=\Delta$ small enough.*

This allows to use the energy estimate (1.7).

We now note that in the proof of Lemma 3.9, with the new thin-slab propagator, $\widehat{\mathcal{G}}_{(z',z)}$, the amplitudes of the operators $\mathcal{D}_1,\dots,\mathcal{D}_4$ only involve the term $a(z',x,\xi)$ instead of both $a(z',x,\xi)$ and $a(z,x,\xi)$ (as $\partial_{z'}((z'-z)\hat{a}_{(z',z)}(x',\xi)) = a(z',x,\xi)$). Thus the proof of Lemma 3.9 does not require any assumption like Assumption 3.7 made in Theorem 3.10 or assumptions of Hölder type regularity on the symbol $a(z,.)$ made in Theorem 3.11. Consequently we obtain

Theorem 3.17. *Let $s\in\mathbb{R}$. Then $\widehat{\mathcal{W}}_{\mathfrak{P},z}$ converges in $L(H^{(s+1)}(\mathbb{R}^n),H^{(s)}(\mathbb{R}^n))$ to the solution operator $U(z,0)$ of the Cauchy problem (1.5)–(1.6) uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to 0 with a convergence rate of order $\frac{1}{2}$:*

$$\|\widehat{\mathcal{W}}_{\mathfrak{P},z} - U(z,0)\|_{(H^{(s+1)},H^{(s)})} \leq C\Delta_{\mathfrak{P}}^{\frac{1}{2}},\quad z\in[0,Z].$$

We may now state the main theorem of this section

Theorem 3.18. *Assume that $a(z,.)$ is in $\mathscr{C}^{0,\alpha}([0,Z],S^1(\mathbb{R}^n\times\mathbb{R}^n))$, i.e., Hölder continuous w.r.t. z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$, in the sense that, for some $0<\alpha<1$*

$$a(z',x,\xi) - a(z,x,\xi) = (z'-z)^\alpha\ \tilde{a}(z',z,x,\xi),\ \ 0\leq z\leq z'\leq Z,$$

or Lipschitz ($\alpha=1$), with $\tilde{a}(z',z,x,\xi)$ bounded w.r.t. z' and z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$. Let $s\in\mathbb{R}$ and $0\leq r<1$. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},z}$ converges to the solution operator $U(z,0)$ of the Cauchy problem (1.5)–(1.6) in $L(H^{(s+1)}(\mathbb{R}^n),H^{(s+r)}(\mathbb{R}^n))$ uniformly w.r.t. z as $\Delta_{\mathfrak{P}}$ goes to 0 with a convergence rate of order $\beta(1-r)$:

$$\|\mathcal{W}_{\mathfrak{P},z} - U(z,0)\|_{(H^{(s+1)},H^{(s+r)})} \leq C\Delta_{\mathfrak{P}}^{\beta(1-r)},\quad z\in[0,Z],$$

with $\beta=\alpha$ for $0<\alpha<\frac{1}{2}$ and $\beta=\frac{1}{2}$ for $\frac{1}{2}\leq\alpha\leq1$. Furthermore, $\mathcal{W}_{\mathfrak{P},z}$ strongly converges to the solution operator $U(z,0)$ in $L(H^{(s+1)}(\mathbb{R}^n),H^{(s+1)}(\mathbb{R}^n))$ uniformly w.r.t. $z\in[0,Z]$.

With the sole assumption of the continuity of the symbol $a(z,.)$ w.r.t. z with values in $S^1(\mathbb{R}^n\times\mathbb{R}^n)$ (Assumption 1.1) the same results hold for the operator $\widehat{\mathcal{W}}_{\mathfrak{P},z}$, with a convergence rate of order $\frac{1-r}{2}$ for the operator convergence in $L(H^{(s+1)}(\mathbb{R}^n),H^{(s+r)}(\mathbb{R}^n))$.

Proof. From energy estimate (1.7) for $s+1$ [8, Theorem 23.1.2] we have

$$\|U(z,0)(u_0)\|_{H^{(s+1)}} \leq C\|u_0\|_{H^{(s+1)}}. \tag{3.30}$$

From Proposition 3.2 we obtain

$$\|\mathcal{W}_{\mathfrak{P},z}(u_0)\|_{H^{(s+1)}} \leq C\|u_0\|_{H^{(s+1)}} \tag{3.31}$$

and thus

$$\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{H^{(s+1)}} \leq C\|u_0\|_{H^{(s+1)}}, \tag{3.32}$$

uniformly w.r.t. $z \in [0, Z]$. The interpolation inequality

$$\|v\|_{H^{(s+r)}} \leq \|v\|_{H^{(s)}}^{1-r} \|v\|_{H^{(s+1)}}^{r}, \quad 0 \leq r \leq 1$$

then yields

$$\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{H^{(s+r)}} \leq C\Delta_{\mathfrak{P}}^{\beta(1-r)}\|u_0\|_{H^{(s+1)}}, \quad 0 \leq r < 1,$$

uniformly w.r.t. $z \in [0, Z]$. For $\widehat{\mathcal{W}}_{\mathfrak{P},z}$ a similar inequality for $\beta = \frac{1}{2}$ is obtained with Assumption 1.1 alone.

Let $u_0 \in H^{(s+1)}$ and let $\varepsilon > 0$. For the strong convergence in $H^{(s+1)}$ we proceed as in Lemma 3.4 and choose $u_1 \in H^{(s+2)}$ such that $\|u_0 - u_1\|_{H^{(s+1)}} \leq \varepsilon$. We then write

$$\begin{aligned}\|\mathcal{W}_{\mathfrak{P},z}(u_0) - U(z,0)(u_0)\|_{H^{(s+1)}} &\leq \|\mathcal{W}_{\mathfrak{P},z}(u_0 - u_1)\|_{H^{(s+1)}} \\ &\quad + \|\mathcal{W}_{\mathfrak{P},z}(u_1) - U(z,0)(u_1)\|_{H^{(s+1)}} + \|U(z,0)(u_0 - u_1)\|_{H^{(s+1)}} \\ &\leq C\varepsilon + C\Delta_{\mathfrak{P}}^{\beta}\|u_1\|_{H^{(s+2)}}\end{aligned}$$

from estimates (3.30) and (3.31) and Theorems 3.10, 3.11 and 3.17, with β as above. This last estimate is uniform w.r.t. $z \in [0, Z]$ and yields the result. ■

A A diagonalization/decoupling of the acoustic wave equation

We give here an overview of [21], which gives a motivation for approximating solutions of the Cauchy problem (1.5)–(1.6), for instance in the context of geophysics.

We first consider the scalar wave equation

$$\left(-\rho^{-1}c^{-2}\partial_t^2 + \sum_{j=1}^{n} \partial_j \rho^{-1} \partial_j\right) u = F, \tag{A.33}$$

as encountered in acoustics, where ρ is the fluid density, and c is the wavespeed. Both these functions are assumed to be independent of time t and to be in $\mathscr{C}^\infty(\mathbb{R}^n)$. We further assume that $0 < \rho_0 \leq \rho(y) \leq \rho_1$ and $0 < c_0 \leq c(y) \leq c_1$, $y \in \mathbb{R}^n$. We denote $z = y_n$ and $x = (y_1, \ldots, y_{n-1})$ and write $p(x, z, D_t, D_x, D_z) = \rho^{-1}c^{-2}D_t^2 - \sum_{j=1}^{n-1} D_j \rho^{-1} D_j - D_z \rho^{-1} D_z$ where $D = \frac{1}{i}\partial$. Its principal symbol is $p_2(t, x, z, \tau, \xi, \zeta) = \rho^{-1}(c^{-2}\tau^2 - |\xi|^2 - \zeta^2)$.

Note that $\tau \neq 0$ in $\mathrm{Char}(p)$. We put (A.33) in a matrix form

$$D_z w(t, x, z) = G(x, z, D_t, D_x) w(t, x, z) + f(t, x, z) \mod \mathscr{C}^\infty, \tag{A.34}$$

$$\text{with } G = \begin{pmatrix} 0 & \Lambda\rho \\ A & 0 \end{pmatrix}, \quad w = \begin{pmatrix} \Lambda u \\ \rho^{-1} D_z u \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ F \end{pmatrix},$$

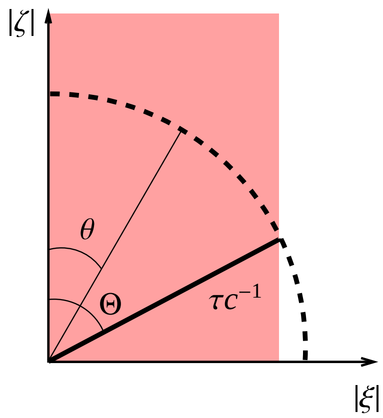

Figure 1: The shaded area corresponds to I_Θ at a given (t, x, z) and a given frequency τ. θ is the propagation angle. The set $\mathrm{Char}(p)$ is represented dotted.

where Λ is a first-order elliptic ψDO, say for instance $|D_{t,x}|$, and

$$A = \rho^{-1} c^{-2} D_t^2 \Lambda^{-1} - \sum_{j=1}^{n-1} D_j \rho^{-1} D_j \Lambda^{-1},$$

with Λ^{-1} denoting a parametrix for Λ.

Following [21], we introduce

$$\begin{aligned} I'_\Theta &= \{(x, z, \tau, \xi) \mid \tau \neq 0, |c(x, z)\tau^{-1}\xi| \leq \sin\Theta\}, \\ I_\Theta &= \{(t, x, z, \tau, \xi, \zeta) \mid (x, z, \tau, \xi) \in I'_\Theta,\ |\zeta| \leq c_0^{-1}|\tau|\}, \end{aligned}$$

where $\Theta \in (0, \frac{\pi}{2})$. The inequality $|\zeta| \leq c(x, z)^{-1}|\tau|$ on $\mathrm{Char}(p)$ explains the condition $|\zeta| \leq c_0^{-1}|\tau|$ above. We choose an angle $\Theta \in (0, \frac{\pi}{2})$ and work in the microlocal region I_Θ assuming that $\mathrm{WF}(u) \subset I_\Theta$. Figure 1 illustrates the set I_Θ at a given (t, x, z) and a given frequency τ. An angle $\theta \in [-\Theta, \Theta]$ corresponds to a propagation angle. Restricting the analysis to I_Θ corresponds to staying away from horizontal propagation. Note that in I_Θ we have $c(x, z)^{-2}\tau^2 - |\xi|^2 > 0$, which is the main purpose of the restriction to such a microlocal region.

In I_Θ, G is a first-order ψDO by Theorem 18.1.35 in [8]. In I_Θ we can follow the method of [25, Chapter IX] (see also [24]) to decouple the up-going and down-going wavefields. We briefly recall the method here. Define $\eta^\pm(x, z, \tau, \xi) = \pm(c(x, z)^{-2}\tau^2 - |\xi|^2)^{\frac{1}{2}}$, which are the two roots of $\det(\eta I_2 - G_1) = 0$ with G_1 the (matrix-)principal symbol of G. The matrix $G_1(x, z, \tau, \xi)$ is diagonalizable and we choose a matrix $V(x, z, \tau, \xi) \in S^0(I'_\Theta)$, invertible, such that VG_1V^{-1} is diagonal; V can be chosen homogeneous of degree 0. If we write $w^{(0)} = V(x, z, D_t, D_x)w$ we obtain

$$\begin{aligned} D_z w^{(0)}(t, x, z) = G^{(0)} w^{(0)}(t, x, z) + f^{(0)}(t, x, z) \quad \mathrm{mod}\ \mathscr{C}^\infty, \\ G^{(0)} = (D_z V)V^{-1} + VGV^{-1} \quad \mathrm{mod}\ \Psi^{-\infty} \text{ in } I_\Theta, \quad f^{(0)} = Vf. \end{aligned}$$

We write $G^{(0)} = G_1^{(0)} + G_0^{(0)}$ with $G_1^{(0)} \in \Psi^1$ in I_Θ and diagonal and $G_0^{(0)} \in \Psi^0$ in I_Θ. We use the notation V^{-1} for a parametrix of V with principal symbol $V(x, z, \tau, \xi)^{-1}$ (an abuse of notations, which will occur below again).

We then write $w^{(1)} = (1 + K^{(1)}(x, z, D_t, D_x))w^{(0)}$, with $K^{(1)} \in \Psi^{-1}$ in I_Θ of the form

$$K^{(1)} = \begin{pmatrix} 0 & K_1^{(1)} \\ K_2^{(1)} & 0 \end{pmatrix}.$$

We then obtain

$$D_z w^{(1)} = G_1^{(0)} w^{(1)} + [K^{(1)}, G_1^{(0)}]w^{(1)} + G_0^{(0)} w^{(1)} + f^{(1)} + R^{(1)} w^{(1)} \mod \mathscr{C}^\infty,$$
$$R^{(1)} \in \Psi^{-1} \text{ in } I_\Theta, \qquad f^{(1)} = (1 + K^{(1)}) f^{(0)},$$

making use of

$$(1 + K^{(1)})G_1^{(0)}(1 + K^{(1)})^{-1} = G_1^{(0)} + [K^{(1)}, G_1^{(0)}](1 + K^{(1)})^{-1}$$

and the fact that $L(1 + K^{(1)})^{-1} - L \in \Psi^{m-1}$ if $L \in \Psi^m$. Lemma 2.1 in [24] shows that $K^{(1)}$ can be chosen so as to have $[K^{(1)}, G_1^{(0)}] + G_0^{(0)}$ diagonal up to an operator in Ψ^{-1} in I_Θ. The procedure goes on by choosing $K^{(2)} \in \Psi^{-2}$ in I_Θ in order to diagonalise the term of order -1, etc. We thus obtain $Q \in \Psi^0$ in I_Θ such that $\tilde{w} = Q^{-1}w$ satisfies

$$D_z \tilde{w} = \tilde{G}\tilde{w} + \tilde{f} \mod \mathscr{C}^\infty, \quad \tilde{f} = Q^{-1} f,$$

with $\tilde{G} = \tilde{G}(x, z, D_t, D_x) \in \Psi^1$ in I_Θ, diagonal up to a regularizing operator

$$\tilde{G} = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}.$$

In [21], Stolk shows that $b_\pm$ can be chosen selfadjoint. This is achieved by first choosing selfadjoint operators with principal symbols equal to $\eta^\pm(x, z, \tau, \xi)$ and then replace $(1 + K^{(i)})$ by $\exp[K^{(i)}]$ in the iteration process described above. Various choices of Q are presented in [21].

We define the set $J_{\Theta+}$ of points $(t_0, x_0, z_0, \tau_0, \xi_0, \zeta_0)$ such that the bicharacteristics associated with b_+, parametrized by z, $(t(z), x(z), \tau(z), \xi(z))$, passing through $(t_0, x_0, \tau_0, \xi_0)$ at $z = z_0$, is such that for all $z \in [0, Z]$, the point $(x(z), z, \xi(z), \tau(z))$ remains in I'_Θ. In other words, with the interpretation given by Figure 1 the propagation angle, $\theta(z)$ along the bicharacteristics should never exceed Θ.

We now choose $0 < \Theta_1 < \Theta_2 < \frac{\pi}{2}$. We choose a real non-negative symbol $c(z, x, \tau, \xi) \in S^1(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1})$ such that $c = 0$ in I_{Θ_1} and elliptic in the complement of I_{Θ_2}. After extending smoothly b_+ outside I_Θ, such that b_+ is real homogeneous of degree 1, we now consider the Cauchy problem

$$(\partial_z - ib_+(z, x, D_t, D_x) + c(z, x, D_t, D_x))v = 0,$$
$$v(0, .) = v_+(0, .),$$

where

$$\tilde{w} = \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = Q^{-1}w = Q^{-1} \begin{pmatrix} \Lambda u \\ \rho^{-1} D_z u \end{pmatrix}.$$

With Assumption (33) and (34) in [21] we obtain that

$$v = v_+ \mod \mathscr{C}^\infty \text{ in } J_{\Theta_1+},$$
$$v = 0 \mod \mathscr{C}^\infty \text{ in the complement of } J_{\Theta_2+}.$$

See [21] and [22] for details. A similar results holds for the other 'one-way' wave operator $\partial_z - ib_- + c$.

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